Numerical methods in Fluid Mechanics.

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Athens Course, Madrid, 14-18 November 2011

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f} + \mu \nabla^2 \mathbf{v}$$

1. Non linear $\mathbf{v} \cdot \nabla \mathbf{v}$ second order ∇^2 partial differential equations.
2. Momentum conservation Equation $\Rightarrow$ 3 scalar equations
3. Mass conservation Equation $\Rightarrow$ 1 scalar equation.
4. 4 Unknonws Velocity
   $\mathbf{v} = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$ and $p(x, y, z, t)$.

Navier-Stokes Equations.

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= 0 \\
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \rho \mathbf{f} + \mu \nabla^2 \mathbf{v}
\end{aligned}
$$

1. Spatial operators: Divergence $\nabla\cdot$, Gradient ∇ and Laplacian ∇^2.
2. Temporal derivatives $\frac{\partial \mathbf{v}}{\partial t}$.

Numerical methods.

1. From the continuum to the discrete world (space and time).
2. Spatial and temporal derivatives are approximated by local values.

Discretization concept.

$$\frac{\partial f(x,y,z,t)}{\partial x} \approx \frac{f(x+\Delta x,y,z,t) - f(x,y,z,t)}{\Delta x}$$

$$\frac{\partial f(x,y,z,t)}{\partial t} \approx \frac{f(x,y,z,t+\Delta t) - f(x,y,z,t)}{\Delta t}$$

Explicit and implicit time discretization. Example viscous term

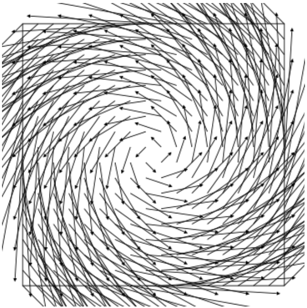
$$
\begin{aligned}
&\textit{Explicit} \\
\nabla^2 \mathbf{v}(x,y,z,t_{n+1}) &= \nabla^2 \mathbf{v}(x,y,z,t_n) \\
&\textit{Implicit} \\
\nabla^2 \mathbf{v}(x,y,z,t_{n+1}) &= \nabla^2 \mathbf{v}(x,y,z,t_{n+1}) \\
&\textit{Semi-implicit} \\
\nabla^2 \mathbf{v}(x,y,z,t_{n+1}) &= \frac{1}{2}(\nabla^2 \mathbf{v}(x,y,z,t_{n+1}) + \nabla^2 \mathbf{v}(x,y,z,t_{n+1}))
\end{aligned}
$$

Exercises.

1. Matlab: numerical derivative.
2. Excel: Laplacian operator.
3. Electromagnetic problem.

Eulerian and Lagrangian description of a fluid.

1. Euler: All fluid variables are vector fields that depend on space and time. $\overrightarrow{v}(\overrightarrow{r}, t)$
2. Lagrangian: Fluids are considered as material moving particles. $\overrightarrow{v}_{particle}(t)$



Lagrangian version of the Navier-Stokes equations.

A velocity field is transported affected by pressure, viscosity and other forces.

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= 0 \\
\rho \frac{d\mathbf{v}}{dt} &= -\nabla p + \rho \mathbf{f} + \mu \nabla^2 \mathbf{v}
\end{aligned}
$$

Let us understand the meaning of the transport operator...

$$\rho \frac{d\mathbf{v}}{dt} = 0$$

$\mathbf{v}$ moves with constant value along its trajectory. Exercise Matlab.

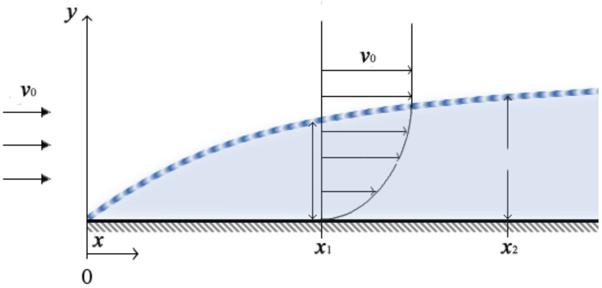
Boundary conditions.

We have to limit the space to solve a problem computationally ⇒ Computational domain. Boundary types

- Walls.
- Inflows
- Outflows
- Periodic

Boundary layer.(Prandtl 1904)

- Layer of fluid in the immediate vicinity of a bounding surface where effects of viscosity are relevant.
- Boundary layer thickness δ (99% of the freestream velocity) growths or decreases depending on the pressure gradient.



If the boundary layer is taken into account in the computation no-slip Boundary Condition is used: $\mathbf{v}_{Wall} = \mathbf{0}$. If boundary layer is not considered $\Rightarrow \mathbf{v}_{Wall} = \mathbf{v}_{Slip}$

Reynolds number.

- Non dimensional version of the Navier-Stokes equations.
- L reference length.
- U reference velocity.
- ρ and μ fluid density and viscosity.

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{f} + \frac{1}{Re} \nabla^2 \mathbf{v}$$

$$Re = \frac{\rho U L}{\mu}$$

Courant number.

- The Courant–Friedrichs–Lewy condition (CFL condition) is a necessary condition for convergence while solving the Navier-Stokes equations.
- Courant number $C = \frac{u_{max}\Delta t}{h}$
  - Δt time step used in the computation.
  - u_{max} Maximum local velocity
  - h Local mesh size.
- CFL condition $C \leq 1$