Chapter 1

THE FINITE ELEMENT METHOD (FEM)

1.1 Galerkin formulation

1.1.1 General procedure

In this section we shall study the Galerkin method, a method which is directly applicable to the Boundary Value Problem (BVP) irrespective of the existence of an equivalent extremal formulation. Let us consider the following linear (Partial Differential Equation) PDE with homogeneous (essencial) Dirichlet boundary conditions on Γ:

$$Lu = f \quad in \quad \Omega \tag{1.1}$$

$$u = 0 \quad in \quad \Gamma \tag{1.2}$$

The solution u is sought in a function space V_o consisting of sufficiently smooth functions satisfying homogeneous Dirichlet conditions on Γ. We assume that the space V_o has a countable basis $\phi_0, \phi_1, \phi_2 ...$, which means that any function $w \in V_o$ can be expressed as an infinite linear combination of basis functions. Formally we can write

$$w = \sum_{j=1}^{\infty} \alpha_j \phi_j \tag{1.3}$$

for any $w \in V_o$. When an arbitrary $w \in V_o$ is substituted into (1.1) the equality is not satisfied. In fact,

$$Lu - f = R \tag{1.4}$$

where $R = R(w)$ is called the residual or error that results from taking w in stead of the solution u. We now try to select an element $u \in V_o$ for which the residual $R(u)$ is zero. The residual is identically zero if its projection on each basis function is equal to zero. So we require that $R(u)$ satisfies:

$$\int_\Omega R(u)\phi_i d\Omega = 0 \quad i = 1, 2, 3, ... \tag{1.5}$$

which implies

$$\int_{\Omega}(Lu - f)\phi_i d\Omega = 0 \quad i = 1, 2, 3, ... \tag{1.6}$$

The order of differentiation in (1.6) can be lowered by performing an integration by parts (Green's formula). Substitution of the boundary conditions leads to an expression of the form

$$a(u, \phi_i) = \int_{\Omega} f\phi_i d\Omega \tag{1.7}$$

which must be satisfied for i = 1,2,... The form $a(.,.)$ is bilinear (see the examples below). Let us remark that by the linearity of (1.7) with respect to ϕ_i, (1.7) is equivalent to

$$\begin{gathered} Find \quad u \in V_o \quad such \quad that \\ a(u, \phi_i) = \int_{\Omega} f\phi_i d\Omega \end{gathered} \tag{1.8}$$

where the functions are called test functions. The Galerkin method now consists in taking a finite dimensional subspace V_{oN} of V_o spanned by N basis functions, say $\phi_1 ..., \phi_N$. The approximate problem can then be defined as

$$\begin{gathered} Find \quad \widetilde{u} \in V_{oN} \quad such \quad that \\ a(\widetilde{u}, \phi) = \int_{\Omega} f\phi d\Omega \quad \forall \phi \in V_{oN} \end{gathered} \tag{1.9}$$

which is equivalent to

$$\begin{gathered} Find \quad \widetilde{u} \in V_{oN} \quad such \quad that \\ a(\widetilde{u}, \phi_i) = \int_{\Omega} f\phi_i d\Omega \quad i = 1, 2, ...N \end{gathered} \tag{1.10}$$

The approximate solution $\widetilde{u}$ being a function in V_{oN} has the form

$$\widetilde{u} = \sum_{j=1}^{N} \alpha_j \phi_j \tag{1.11}$$

Substitution of (1.11) into (1.10) leads to the following system of linear algebraic equations for $\alpha_1, ..., \alpha_N$:

$$\sum_{j=1}^{N} a(\phi_j, \phi_i) = \int_{\Omega} f\phi_i d\Omega \quad i = 1, 2, ...N \tag{1.12}$$

It may be clear that for the actual construction of the basis $\phi_1, ..., \phi_N$ functions the FEM can be used most efficiently.

Problem (1.8) is known as the weak (or variational) formulation of problem (1.1) and it is obtained starting from the BVP formulation. In order to prove that both formulations, the BVP and the variational problem, are equivalent, we still have to prove that a solution of the variational problem is actually a solution of the original BVP.

When, instead of homogeneous Dirichlet conditions, the solution must satisfy inhomogeneous Dirichlet conditions $u = g_o$ on Γ, we write u as

$$u = G_o + \sum_{j=1}^{\infty} \alpha_j \phi_j \tag{1.13}$$

where G_o is such that $G_o|_\Gamma = g_0$ and $\phi_1, ..., \phi_N$ vanish on Γ. The (equivalent) weak formulation then will be

$$\begin{aligned} &Find \quad u \in V_{go} \quad such \quad that \\ &a(u, \phi) = \int_\Omega f \phi d\Omega \quad \forall \phi \in V_o \end{aligned} \tag{1.14}$$

where V denotes the set of functions of the form (1.13):

$$V_{go} = V_o + G_o \tag{1.15}$$

The approximate problem is formulated as follows:

$$\begin{aligned} &Find \quad \widetilde{u} \in V_{g_o N} \quad such \quad that \\ &a(\widetilde{u}, \phi) = \int_\Omega f \phi d\Omega \quad \forall \phi \in V_{oN} \end{aligned} \tag{1.16}$$

where $V_{goN} = G_o + V_{ON}$.

1.1.2 1D Poisson equation; Homogeneous boundary conditions

Consider the following Poisson equation in 1D

$$-\frac{d^2u}{dx^2} = f \quad on \quad (0,1) \tag{1.17}$$

with boundary conditions

$$u(0) = 0 \quad \frac{du}{dx}(1) = 0 \tag{1.18}$$

We consider the function space V of sufficiently smooth functions that vanish at $x = 0$. Let $\phi \in V$ be arbitrary. It follows from (1.17) that

$$-\int_0^1 \frac{d^2u}{dx^2} \phi dx = \int_0^1 f \phi dx \tag{1.19}$$

Integration by parts of the left hand side gives:

$$\int_0^1 \frac{du}{dx} \frac{d\phi}{dx} dx - [\frac{du}{dx} \phi]_0^1 = \int_0^1 f \phi dx \tag{1.20}$$

Since $u(0) = 0$ and $\frac{du}{dx}(1) = 0$, (1.20) reduces to

$$\int_0^1 \frac{du}{dx} \frac{d\phi}{dx} dx = \int_0^1 f \phi dx \tag{1.21}$$

The variational formulation of problem (1.20), (1.20) reads now

$$
\begin{aligned}
&Find \quad u \in V \quad such \quad that \\
&a(u, \phi_i) = \int_0^1 f\phi dx \quad \forall \phi \in V
\end{aligned} \tag{1.22}
$$

with $a(u, \phi) = \int_0^1 \frac{du}{dx}\frac{d\phi}{dx}dx$.

In the space V we choose a finite number of basis functions $\phi_1, ..., \phi_N$. When the function space spanned by these basis functions is denoted by V_N the approximate variational formulation reads:

$$
\begin{aligned}
&Find \quad \widetilde{u} \in V_N \quad such \quad that \\
&a(\widetilde{u}, \phi_i) = \int_0^1 f\phi dx \quad \forall \phi \in V_N
\end{aligned} \tag{1.23}
$$

Writing $\widetilde{u}$ in the form

$$
\widetilde{u} = \sum_{j=1}^{N} \alpha_j \phi_j \tag{1.24}
$$

then, (1.23) is equivalent to the following system of equations for $\alpha_1, ..., \alpha_N$.

$$
\sum_{j=1}^{N} \alpha_j \int_0^1 \phi_j \phi_i dx = \int_0^1 f\phi_i dx \quad i = 1, 2, ...N \tag{1.25}
$$

The FEM can be used to choose the basis functions. We subdivide the interval $(0, l)$ into N subregions (subintervals), we choose the extremities of the subintervals as nodal points $0 = x^0 < x^1 < ...x^N = 1$. The basis functions $\phi_i \quad i = 1, 2, ..., N$ are then completely determined by the following three properties

1. $\phi_i(x^j) = \delta_{ij} \quad i, j = 1, ...N$.
2. ϕ_i is linear on each subinterval
3. ϕ_i is continuous on $[0, 1]$.

Then

$$
\widetilde{u}(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x) \tag{1.26}
$$

and the system of linear equations for $\widetilde{u}_l, ..., \widetilde{u}_N$ becomes

$$
\sum_{j=1}^{N} \widetilde{u}_j \int_0^1 \phi_j \phi_i dx = \int_0^1 f\phi_i dx \quad i = 1, 2, ...N \tag{1.27}
$$

1.1.3 1D Poisson equation; non-homogeneous boundary conditions.

Consider the Poisson equation

$$-\frac{d^2u}{dx^2} = f \quad on \quad (0,1) \tag{1.28}$$

with boundary conditions

$$u(0) = g_0 \quad \frac{du}{dx}(1) = g_1 \tag{1.29}$$

We define the variable change $\widehat{u} = u - u_\Gamma$, where u_Γ is defined as:

$$u_\Gamma = g_0 \quad if \quad x = 0 \tag{1.30}$$

$$u_\Gamma = 0 \quad rest \tag{1.31}$$

Now the problem can be written as:

$$-\frac{d^2(\widehat{u} + u_\Gamma)}{dx^2} = f \quad on \quad (0,1) \tag{1.32}$$

with the boundary conditions

$$\widehat{u}(0) = 0 \quad \frac{d\widehat{u}}{dx}(1) = g_1 \tag{1.33}$$

This new problem is an homogeneous one and it can be solved as we already know. The solution comes from the solution of the system for $\widehat{u}_l, ..., \widehat{u}_N$:

$$\sum_{j=1}^{N} \widehat{u}_j \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx = \int_0^1 f\phi_i dx - g_0 \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx + g_1\phi_i \quad i = 1, 2, ...N \tag{1.34}$$

This system of equations differs only slightly from the one obtained with homogeneous Dirichlet-Neumann boundary conditions. Only the terms $-g_0 \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx$ and $g_1\phi_i$ have been added to the right hand side.

To obtain the final solution we have to replace the original function $u = \widehat{u} + u_\Gamma$, so:

$$u = \sum_{j=1}^{N} \widehat{u}_j\phi_j + u_\Gamma \tag{1.35}$$

which means adding g_0 locally at $x = 0$.

It is important to remark that the solution of (1.34) means the solution of a linear system. Calling:

$$R_{ij} = \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx \tag{1.36}$$

$$b_i = \int_0^1 f\phi_i dx - R_{ij}g_0 + g_1\phi_i \tag{1.37}$$

$$\sum_{j=1}^{N} R_{ij}\widehat{u}_j = b_i \tag{1.38}$$

1.2 Construction of a basis

In this chapter we shall construct finite elements in the 1D case and also the construction of quadratic and cubic 1D elements will be discussed. To facilitate the construction of these elements we shall introduce the notion of barycentric coordinates. Next we introduce coordinate and element transformation and the notion of isoparametric finite element.

1.2.1 Linear and quadratic functions in 1D

Let us consider an arbitrary interval $e = [x_1, x_2] \subset [0, 1]$. We define $\forall x \in e$ the following functions λ_1 and λ_2 from $[x_1, x_2]$

$$\lambda_1 = \frac{x_2 - x}{x_2 - x_1} \quad \lambda_2 = \frac{x - x_1}{x_2 - x_1} \tag{1.39}$$

These functions have the following properties:

1. X is linear on e, i = 1,2.
2. $\lambda_i(x_j) = \delta_{ij}$.
3. $\lambda_1(x) + \lambda_2(x) = 1 \quad \forall x \in e$.

The relation of the functions λ_1 and λ_2 to the 1D piecewise linear basis functions is evident. In fact, the basis function ϕ_i corresponding to $x_i, i = 1, 2$ satisfies:

1. ϕ_i is linear on each subinterval.
2. ϕ_i is continuous on [0,1].
3. $\phi_i(x_j) = \delta_{ij} \quad i, j = 1, 2$.

so that the restriction of ϕ_i to e is precisely the function λ_i(See Fig.1.1 (a) and (b)) For any point $x \in e$ we can calculate the values of the functions X and h2. The pair $\{\lambda_1(x), \lambda_2(x)\}$ is called the barycentric coordinates of the point x on e with respect to the points x and x. The 1D linear finite element is now defined as: (1.) a subdivision of [0,1] into subintervals, (2.) on each subinterval we choose two nodal points: the end points x_1 and x_2 of the subinterval, (3.) on each subinterval we define for each nodal point its basis function: on $[x_1, x_2]$ for instance

$$\phi_1 = \lambda_1 \quad \phi_2 = \lambda_2 \tag{1.40}$$

The function space spanned by ϕ_1 and ϕ_2 on e is denoted by $P_1(e)$, and contains all polynomials of degree ≤ 1 in x. Generally we denote by $P_k(e)$, k non-negative integer, $e \subset \mathbb{R}$ the function space of polynomials of degree $< k$ in $x_1, x_2, ..., x_n$. In other words, $P_k(e)$ is the function space spanned by $x_1^{k_1} x_2^{k_2} x_n^{k_n}$

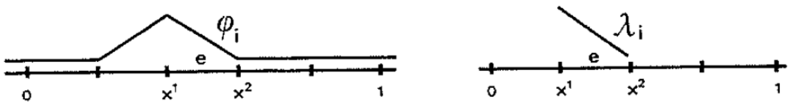

Figure 1.1: (a) Basis function ϕ_i (b)The function λ_i

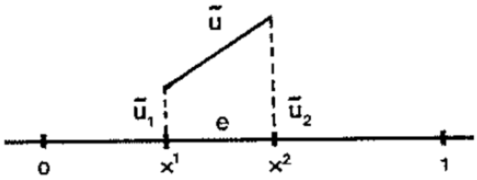

Figure 1.2: Linear shape function $\widetilde{u}$ on e

with $k_i > 0$, $i = 1, ..., n$, $k_1 + k_2 + ... + k_n < k$. It follows that the shape function $\widetilde{u}$ takes the following form on e (see Fig.1.2)

$$\widetilde{u}(x) = \widetilde{u}_1\phi_1(x) + \widetilde{u}_2\phi_2(x) = \widetilde{u}_1\lambda_1(x) + \widetilde{u}_2\lambda_2(x) \tag{1.41}$$

Notice that, since the basis functions are linear on e, their derivatives are constant:

$$\begin{aligned} \frac{d\phi_1}{dx} &= \frac{d\lambda_1}{dx} = \frac{-1}{x^2 - x^1} \\ \frac{d\phi_2}{dx} &= \frac{d\lambda_2}{dx} = \frac{1}{x^2 - x^1} \end{aligned} \tag{1.42}$$

The barycentric coordinates $\{\lambda_1, \lambda_2\}$ can be used to define higher order basis functions in a very efficient way. Let us consider 1D basis functions that are quadratic on each subinterval. We assume that the interval [0,1] is subdivided into subintervals. To define uniquely a quadratic function on a subinterval $e = [x_1, x_2]$ we must fix its values in three different points. For this we choose on each subinterval e the following nodal points: the two end points which we call x^1 and x^2 and the mid-point $x^{12}=\frac{1}{2}(x^1 + x^2)$ (See Fig.1.3)

Let $\{\lambda_1, \lambda_2\}$ be the barycentric coordinates of a point $x \in e$ with respect to x_1, x_2. The function λ_1 is linear and $\lambda_i(x^j) = \delta_{ij},\ i, j = 1, 2$. Moreover $\lambda_1(x^{12}) = \lambda_2(x^{12}) = \frac{1}{2}$. Next we notice that $\lambda_1(x^2) = 0$ and $\lambda_1(x^{12}) - \frac{1}{2} = 0$. Hence the function $\lambda_1(x)(\lambda_1(x) - \frac{1}{2})$ vanishes at $x = x^2$ and $x = x^{12}$ and is quadratic since both λ_1 and $\lambda_1 - \frac{1}{2}$ are linear. For $x = x^1$ the function takes the value $\lambda_1(x^1)(\lambda_1(x^1) - \frac{1}{2}) = 1(1 - \frac{1}{2}) = \frac{1}{2}$.
From this we deduce that the function

$$\phi_1(x) = 2\lambda_1(x)(\lambda_1(x) - \frac{1}{2}) \tag{1.43}$$

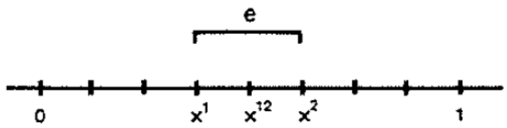

Figure 1.3: Element e with nodal points x^1, x^2 and x^{12}

Figure 1.4: Quadratic basis functions

is the quadratic basis function on e corresponding to point x^1.
Similarly we find that

$$\phi_2(x) = 2\lambda_2(x)(\lambda_2(x) - \frac{1}{2}) \tag{1.44}$$

is the quadratic basis function on e corresponding to point x^2.
For point x_{12} we remark that $\lambda_1(x^2) = 0$, $\lambda_2(x^1) = 0$ and that $\lambda_1(x^{12})\lambda_2(x^{12}) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$ which makes that the function

$$\phi_{12}(x) = 4\lambda_1(x)\lambda_2(x) \tag{1.45}$$

is the quadratic basis function on e corresponding to point x^{12}.
The three basis functions are depicted in Fig.1.4

The function space spanned by ϕ_1, ϕ_2 and ϕ_{12} on e is denoted by $P_2(e)$ and consist of all polynomials of degree ≤ 2 in x_1. The quadratic shape function takes now the following parabolic form on e See Fig.1.5.

$$\begin{aligned} \widetilde{u}(x) &= \widetilde{u}_1\phi_1(x) + \widetilde{u}_2\phi_2(x) + \widetilde{u}_{12}\phi_{12}(x) \\ &= \widetilde{u}_1\lambda_1(x)(2\lambda_1(x) - 1) + \widetilde{u}_2\lambda_2(x)(2\lambda_2(x) - 1) + 4\widetilde{u}_{12}\lambda_1(x)\lambda_2(x) \end{aligned} \tag{1.46}$$

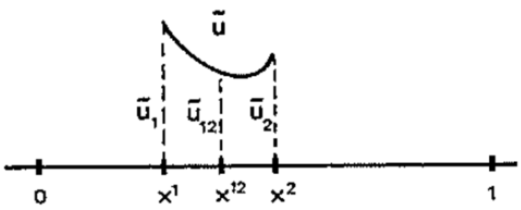

Figure 1.5: Quadratic shape function $\widetilde{u}$ on e

1.3 Triangular basis functions in 2D

1.3.1 Barycentric coordinates

In this section we shall use barycentric coordinartes to define linear, quadratic and extended quadratic basis functions in 2D. Let Ω be a 2D region which is subdivided into a finite number of triangles $e_k, k = 1, 2, ..., K$, satisfying the properties (falta referencia). A linear function on a triangle e is completely determined by its values in-three non-collinear points, for example the vertices $\underline{x}^1, \underline{x}^2$ and $\underline{x}^3$ with coordinates $\underline{x}^1 = (x_1^1, x_2^1)$, $\underline{x}^2 = (x_1^2, x_2^2)$, $\underline{x}^3 = (x_1^3, x_2^3)$ We shall now define three functions $\lambda_i = \lambda(\underline{x}), i = 1, 2, 3$ on e by the following requirements

1. $\lambda_i(\underline{x}^j) = \delta_{ij}$.

2. λ_i is linear on e

These functions λ_i can be calculated as follows: Take for instance λ_1.Since λ_1 is linear it takes the following form:

$$\lambda_i = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \qquad \underline{x} = (x_1^1, x_2^1)$$

Moreover we have

$$\begin{aligned} \lambda_1(\underline{x}^1) &= \alpha_0 + \alpha_1 x_1^1 + \alpha_2 x_2^1 \\ \lambda_1(\underline{x}^2) &= \alpha_0 + \alpha_1 x_1^2 + \alpha_2 x_2^2 \\ \lambda_1(\underline{x}^3) &= \alpha_0 + \alpha_1 x_1^3 + \alpha_2 x_2^3 \end{aligned} \tag{1.47}$$

System (1.47) can be solved for α_0, α_1 and α_2; we find

$$\alpha_0 = \frac{x_1^2 x_2^3 - x_1^3 x_2^2}{\Delta} \quad \alpha_1 = \frac{x_2^2 - x_2^3}{\Delta} \quad \alpha_2 = \frac{x_1^3 - x_1^2}{\Delta}$$

with

$$\begin{aligned} \Delta &= (x_1^3 - x_1^2)(x_2^1 - x_2^2) - (x_2^2 - x_2^3)(x_1^2 - x_1^1) \\ &= \pm\text{twice the area of } e \end{aligned}$$

Figure 1.6: Triangle e with particular points

Similar expressions can be found for λ_1 and λ_3. We easily verify that the following relation holds

$$\lambda_1(\underline{x}) + \lambda_2(\underline{x}) + \lambda_3(\underline{x}) = 1 \qquad \text{for all } \underline{x} \in e \tag{1.48}$$

For any point $\underline{x} \in e$ we can calculate the values of the functions λ_1,λ_2 and λ_3. Now, similar to the definition in the 1D case, we call the triple $\{\lambda_1(\underline{x}), \lambda_2(\underline{x}), \lambda_3(\underline{x})\}$ the barycentric coordinates of the point $\underline{x} \in e$ with respect to the vertices $\underline{x}^1, \underline{x}^2, \underline{x}^3$. Let us calculate the barycentric coordinates of some particular points in a triangle e (see Fig.1.6).

$$\begin{aligned}
\underline{x}^{12} &= \frac{1}{2}(\underline{x}^1 + \underline{x}^2) \quad \text{mid-point of segment } [\underline{x}^1, \underline{x}^2] \\
\underline{x}^{13} &= \frac{1}{2}(\underline{x}^1 + \underline{x}^3) \quad \text{mid-point of segment } [\underline{x}^1, \underline{x}^3] \\
\underline{x}^{23} &= \frac{1}{2}(\underline{x}^2 + \underline{x}^3) \quad \text{mid-point of segment } [\underline{x}^2, \underline{x}^3] \\
\underline{x}^{123} &= \frac{1}{3}(\underline{x}^1 + \underline{x}^2 + \underline{x}^3) \quad \text{barycentre of } e
\end{aligned}$$

$$\begin{aligned}
\underline{x}^1 &\to \{1, 0, 0\} \\
\underline{x}^2 &\to \{0, 1, 0\} \\
\underline{x}^3 &\to \{0, 0, 1\} \\
\underline{x}^{12} &\to \{\frac{1}{2}, \frac{1}{2}, 0\} \\
\underline{x}^{13} &\to \{\frac{1}{2}, 0, \frac{1}{2}\} \\
\underline{x}^{23} &\to \{0, \frac{1}{2}, \frac{1}{2}\} \\
\underline{x}^{123} &\to \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}
\end{aligned}$$

Figure 1.7: λ-relations for some segments

Notice also that points on the following segments satisfy a relation expressed in the barycentric coordinates:

$$\begin{aligned}
\underline{x} \in [\underline{x}^1, \underline{x}^2] \quad &\Leftrightarrow \quad \lambda_3(\underline{x}) = 0 \\
\underline{x} \in [\underline{x}^1, \underline{x}^3] \quad &\Leftrightarrow \quad \lambda_2(\underline{x}) = 0 \\
\underline{x} \in [\underline{x}^2, \underline{x}^3] \quad &\Leftrightarrow \quad \lambda_1(\underline{x}) = 0 \\
\underline{x} \in [\underline{x}^{12}, \underline{x}^{13}] \quad &\Leftrightarrow \quad \lambda_1(\underline{x}) = \frac{1}{2} \\
\underline{x} \in [\underline{x}^{12}, \underline{x}^{23}] \quad &\Leftrightarrow \quad \lambda_1(\underline{x}) = \frac{1}{2} \\
\underline{x} \in [\underline{x}^{13}, \underline{x}^{23}] \quad &\Leftrightarrow \quad \lambda_3(\underline{x}) = \frac{1}{2}
\end{aligned}$$

The situation is sketched in Fig.1.7

Using these barycentric coordinates we shall now construct linear,quadratic and extended quadratic basis functions.

1.3.2 Linear finite element

For the construction of piecewise linear basis functions we take all the vertices of the triangles as nodal points:$\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^N$. A piecewise linear basis function ϕ_i corresponding to nodal point $\underline{x}^i$ is such that

1. $\phi_i(\underline{x}^j) = \delta_{ij} \qquad i, j = 1, 2, \ldots, N$
2. ϕ_i is linear on each e_k
3. ϕ_i is continuous on $\overline{\Omega}$

We plainly verify that the basis function ϕ_i is identically zero on those triangles for which $\underline{x}^1$ is no vertex. Thus, let a triangle e with vertex $\underline{x}^i$ be given. Introduce a local numbering of the vertices of e (see Fig.1.8

The question now is: what is the shape on e of the basis functions ϕ_i corresponding to the points $\underline{x}^i$, i=1,2,3. The function ϕ_i is linear on e and satisfies $\phi_i(\underline{x}^j) = \delta_{ij}$, i,j=1,2,3. So we see inmediately that

$$\phi_1 = \lambda_1, \quad \phi_2 = \lambda_2, \quad \phi_3 = \lambda_3 \tag{1.49}$$

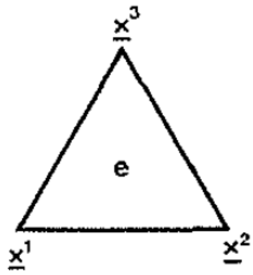

Figure 1.8: Triangle e with nodal points $\underline{x}^1$, $\underline{x}^2$, $\underline{x}^1$

where $\{\lambda_1(\underline{x}), \lambda_2(\underline{x}), \lambda_3(\underline{x})\}$ denotes the barycentric coordinates of $\underline{x}$ with respect to the points $\underline{x}^1$,$\underline{x}^2$,$\underline{x}^3$.The function space spanned by ϕ_1, ϕ_2, ϕ_3 on e (the shape functions on e) is termed $P_1(e)$ and is exactly the collection of polynomials on e of degree ≤ 1 in x_1 and x_2.

1.3.3 Quadratic finite element

A piecewise quadratic basis function ϕ has in 2D the general form:

$$\phi(\underline{x}) = \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 + \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6$$

This implies that on each triangle ϕ must be specified by six values to determine the parameters $\alpha_1, \alpha_2, \ldots, \alpha_6$. On each triangle we choose six nodal points: the three vertices and the three mid-points of sides. With local renumbering we have the situation of Fig.1.9.

We shall determine the shape of the basis functions $\phi_1, \phi_2, \phi_3, \phi_{12}, \phi_{13}, \phi_{23}$ corresponding to the points $\underline{x}^1, \underline{x}^2, \underline{x}^3, \underline{x}^{12}, \underline{x}^{13}, \underline{x}^{23}$ respectively.
Take for instance ϕ_1 which vanishes in the nodal points $\underline{x}^2, \underline{x}^3, \underline{x}^{12}, \underline{x}^{13}, \underline{x}^{23}$ and equals unity at $\underline{x}^1$. For all $\underline{x} \in [\underline{x}^2, \underline{x}^3]$ we have $\lambda_1(\underline{x}) = 0$, for all $\underline{x} \in [\underline{x}^{12}, \underline{x}^{13}]$ we have $\lambda_1(\underline{x}) = \frac{1}{2}$. Consequently the function $\lambda_1(\underline{x})(\lambda_1(\underline{x}) - \frac{1}{2})$ vanishes at $\underline{x}^2, \underline{x}^3, \underline{x}^{12}, \underline{x}^{13}, \underline{x}^{23}$ and equals $\frac{1}{2}$ at $\underline{x}^1$. The basis function ϕ_1 is thus defined by

$$\phi_1 = \lambda_1(2\lambda_1 - 1) \tag{1.50}$$

In the same way we obtain

$$\phi_2 = \lambda_2(2\lambda_2 - 1) \tag{1.51}$$

$$\phi_3 = \lambda_3(2\lambda_3 - 1) \tag{1.52}$$

For the basis function ϕ_{12} we notice that segment $[\underline{x}^1, \underline{x}^3]$ satisfies $\lambda_2(\underline{x}) = 0$ and that $[\underline{x}^2, \underline{x}^3]$ satisfies $\lambda_1(\underline{x}) = 0$. The function $\lambda_1(\underline{x})\lambda_2(\underline{x})$ thus vanishes at $\underline{x}^1, \underline{x}^2, \underline{x}^3, \underline{x}^{13}, \underline{x}^{23}$ and equals $\frac{1}{4}$ at $\underline{x}^{12}$. Hence ϕ_{12} is defined by

$$\phi_{12} = 4\lambda_1\lambda_2 \tag{1.53}$$

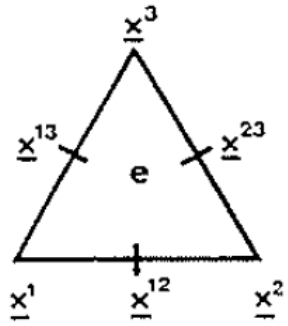

Figure 1.9: Triangle e with six nodal points

Similarly we have

$$\phi_{13} = 4\lambda_1\lambda_3 \tag{1.54}$$

$$\phi_{23} = 4\lambda_2\lambda_3 \tag{1.55}$$

The function space spanned by $\phi_1, \phi_2, \phi_3, \phi_{12}, \phi_{13}, \phi_{23}$ is called $P_2(e)$ and is precisely the space of polynomials of degree ≤ 2 in x_1 and x_2