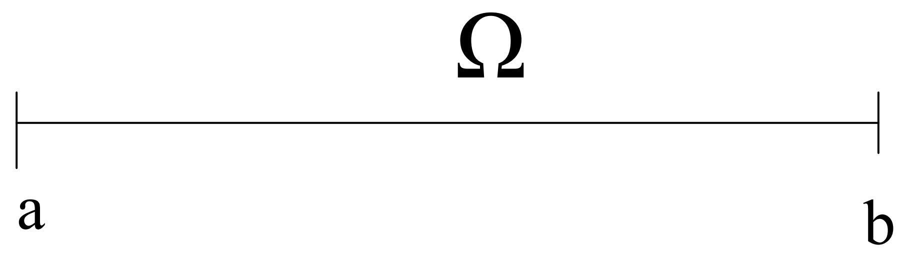
The Poisson problem

Let us solve the following Poisson problem in 1D:

$$-\frac{d^2u}{dx^2} = f(x) \qquad \textbf{in} \quad \Omega$$

$$u(x=a) = u(x=b) = 0$$



In our case:

$$-\frac{d^2u}{dx^2} = \left(\frac{\Pi}{L}\right)^2 sen\left(\frac{\Pi X}{L}\right) \qquad \textbf{in} \quad \Omega$$

$$u(a) = u(b) = 0$$

FEM Discretization

- Let us define a Hilbert space $L^2(\Omega)$
  - The elements of the vectorial space are functions.
  - The square of the functions have a finite quadrature.
  - The scalar product is defined as

$$\langle f, g \rangle = \int_{\Omega} f(x) \cdot g(x) \cdot dx$$

- Where f and g are $L^2(\Omega)$ vectors.

FEM Discretization

- Let us define an approximated Hilbert space

  – Vectors are polynomia which depend on “h”-parameters

  – Notation → $L_h^2(\Omega)$

FEM Discretization

- The FEM gives a solution contained in $L_h^2(\Omega)$

$$\underline{L^2(\Omega)} \qquad \qquad \underline{L_h^2(\Omega)}$$

$$f \qquad \to \qquad f_h = \sum_{i=1}^{h} f_i \cdot \Phi_i$$

FEM Discretization

- Φ_j =Basis functions.
  - Let x_i be a node in the dominium Ω $\left(x^i{}_1, x^i{}_2\right)$

Defined as:

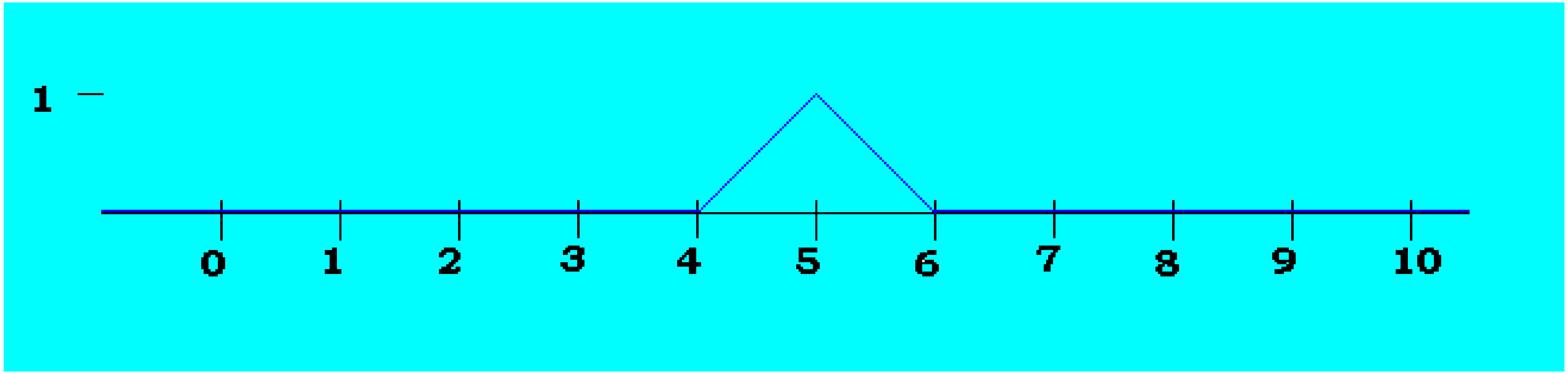
$$\Phi_j\left(x_i\right) = \delta_i^j$$

  - Linear and continous

FEM Discretization

$1D \quad \Omega = [0,10]$

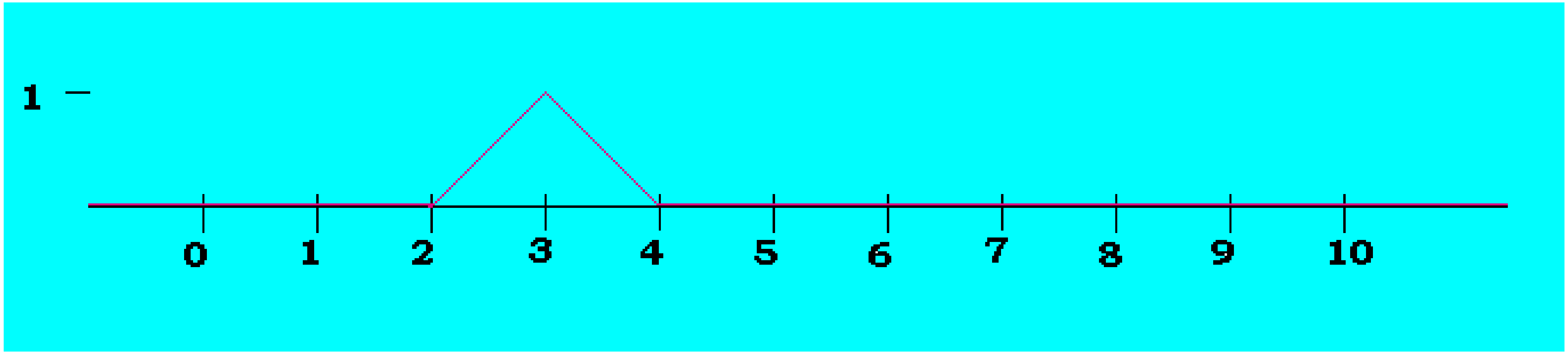
$$\Phi_5(\underline{x}) \quad verify \begin{cases} \Phi_5(\underline{x}^5) = 1 \\ \Phi_5(\underline{x}^j)_{j \neq 5} = 0 \end{cases}$$



FEM Discretization

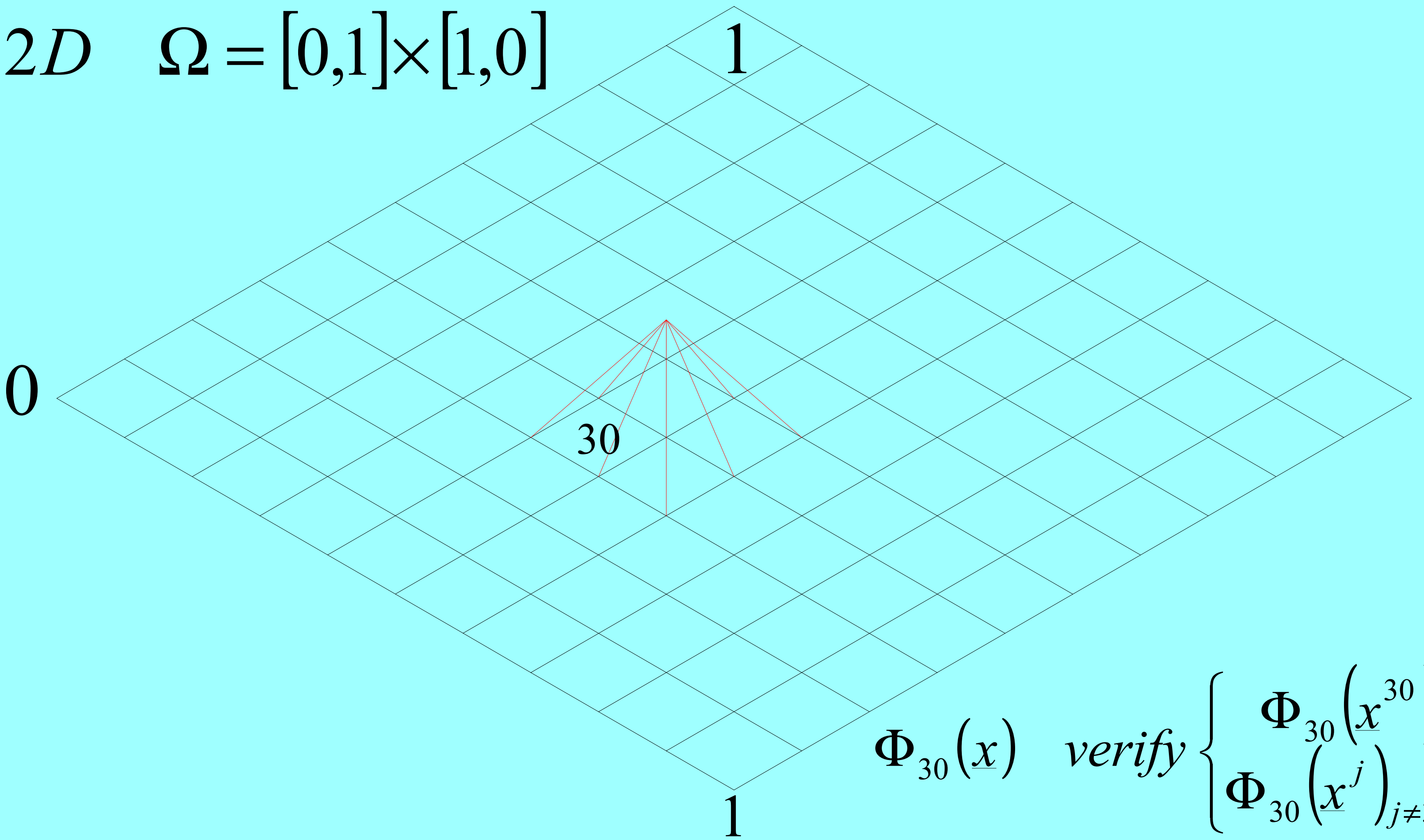
$1D \quad \Omega = [0,10]$

$$\Phi_3(\underline{x}) \quad verify \begin{cases} \Phi_3(\underline{x}^3) = 1 \\ \Phi_3(\underline{x}^j)_{j \neq 3} = 0 \end{cases}$$



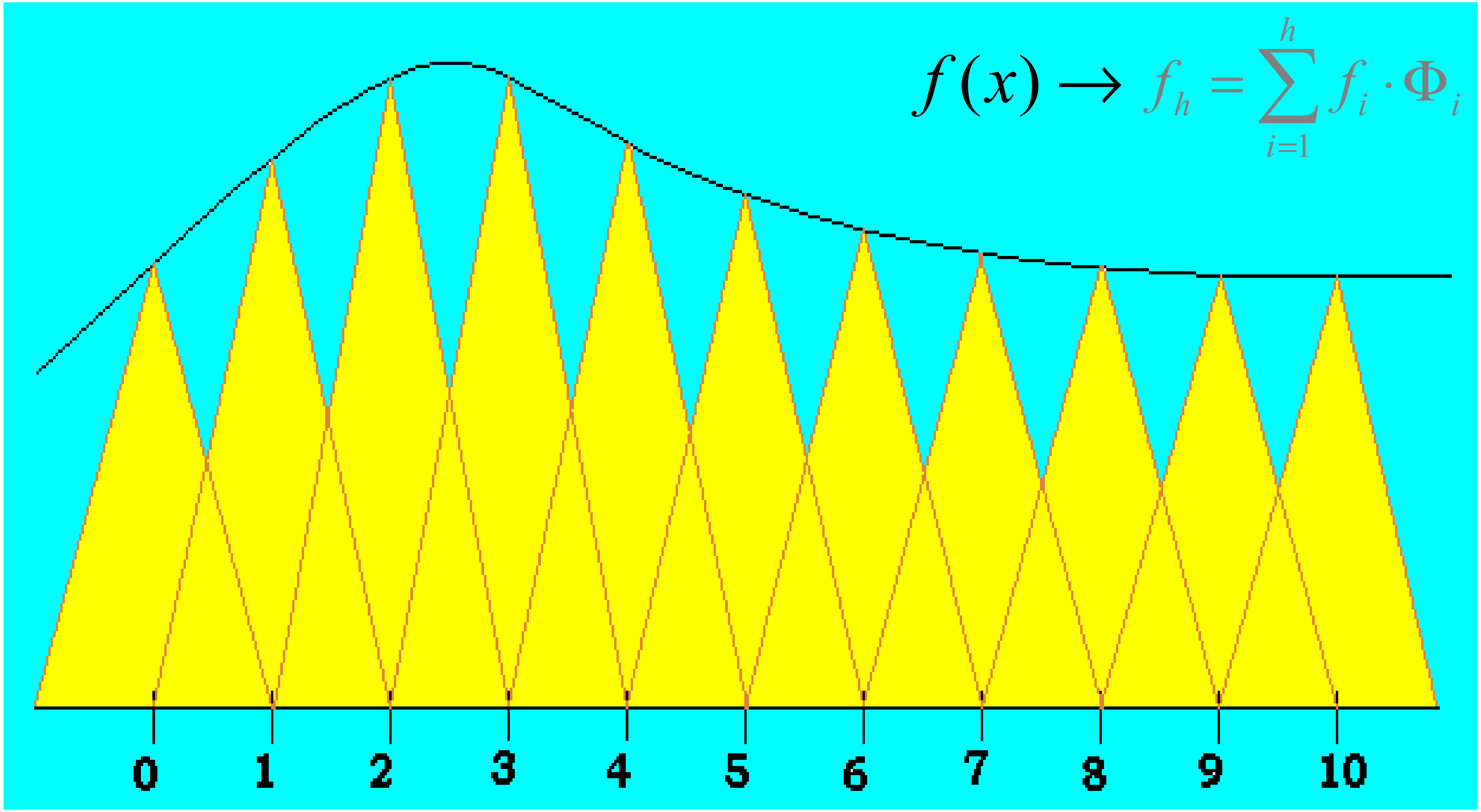
FEM Discretization

$2D \quad \Omega = [0,1] \times [1,0]$



$$\Phi_{30}(\underline{x}) \quad \textit{verify} \begin{cases} \Phi_{30}\left(\underline{x}^{30}\right) = 1 \\ \Phi_{30}\left(\underline{x}^{j}\right)_{j \neq 30} = 0 \end{cases}$$

FEM Discretization



Solution

$$-\frac{d^2u}{dx^2} = f \qquad \phi_j(a) = \phi_j(b) = 0$$

$$\int_a^b -\frac{d^2u}{dx^2}\cdot\Phi_j dx = \int_a^b f\cdot\Phi_j dx$$

Green's Theorem
(Integration by parts)

$$\frac{du}{dx}\cdot\Phi_j(b) - \frac{du}{dx}\cdot\Phi_j(a) - \int_a^b \frac{d\Phi_j}{dx}\cdot\frac{du}{dx}\cdot dx = -\int_a^b f\cdot\Phi_j dx$$

$$\int_a^b \frac{d\Phi_j}{dx}\frac{du}{dx}dx = \int_a^b f\cdot\Phi_j dx$$

Solution

$$u_h = \sum_i u_i \cdot \Phi_i \qquad f_h = \sum_i f_i \cdot \Phi_i$$

$$\sum_i u_i \cdot \int_\Omega \nabla\Phi_j \cdot \nabla\Phi_i = \sum_i f_i \cdot \int_\Omega \Phi_i \cdot \Phi_j$$

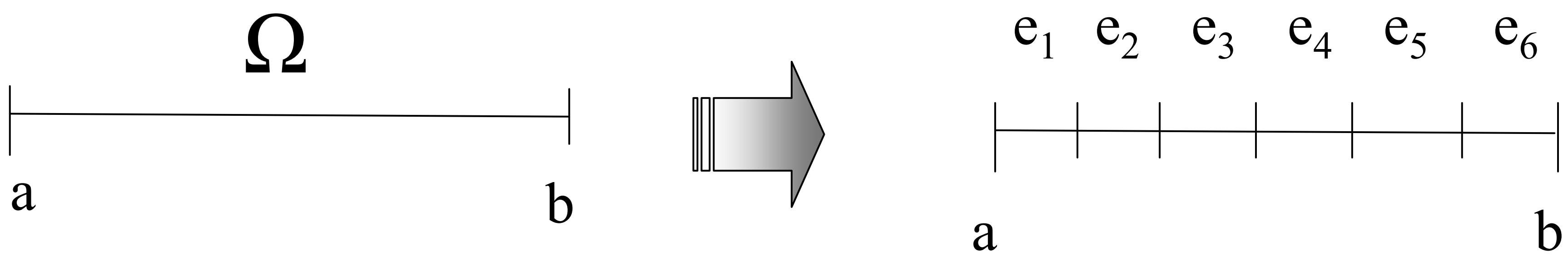
Stiff matrix R(i,j)

Mass matrix M(i,j)

Linear System:

$$R_{i,j} \cdot u_i = f_i \cdot M_{i,j}$$

Matrix calculation



$$[a,b] \Rightarrow [a,b] = \bigcup_{k=1}^{M} e_k$$

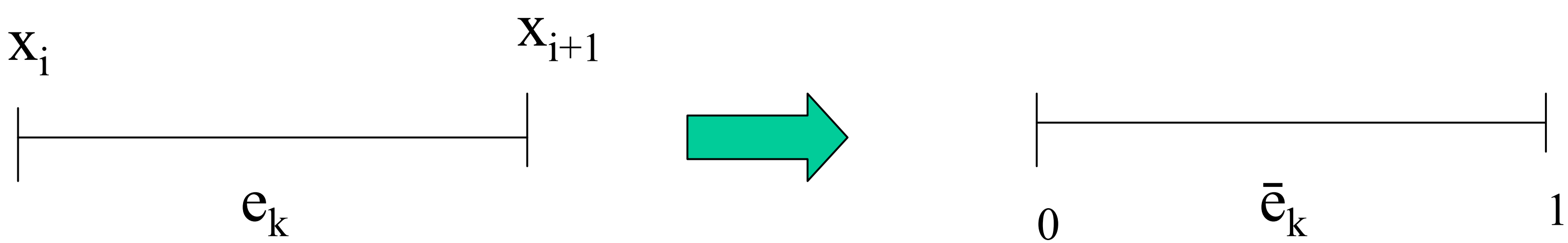
Matrix calculation

Stiff matrix

$$\int_a^b \frac{d\Phi_j}{dx}\frac{d\Phi_i}{dx}dx = \sum_{k=1}^{M}\int_{e_k}\frac{d\Phi_j}{dx}\frac{d\Phi_i}{dx}dx$$

Mapping

$$\int_{e_k} f(x)dx \quad \Rightarrow \quad \int_{\hat{e}_k} f(r)\left|Jac\right| dr$$

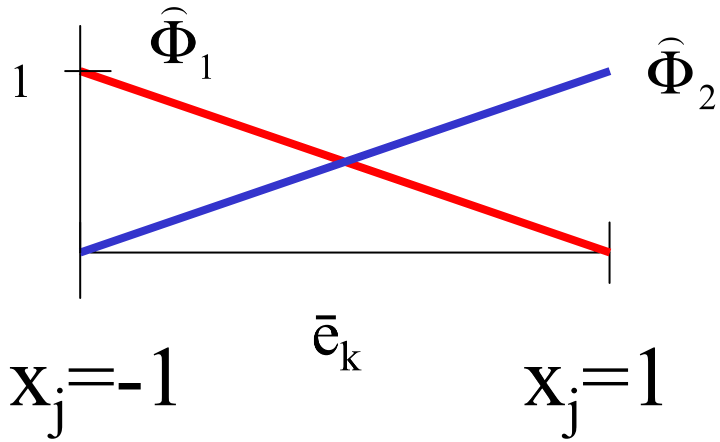


Element k

Reference element

Basis Functions(Linear case)

$$\hat{\Phi}_1 = \frac{1-r}{2}$$



$$\hat{\Phi}_2 = \frac{1+r}{2}$$

$$\hat{\Phi}_i(x_j) = \delta_{ij}$$

$$x(r) = \sum_i x_i \hat{\Phi}_i(r)$$

Gauss quadratures

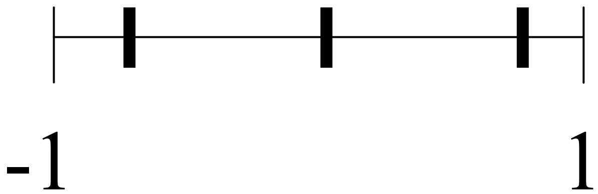
$$\sum_{k=1}^{M} \int_{e_k} \left(\frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} \right) de_k$$

$$\int_{\widehat{e}_k} f(x)dx \quad \Rightarrow \quad \sum_{l}^{ngp} w_l \times f(x_l)$$

$$\sum_{k=1}^{M} \int_{e_k} \left(\frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} \right) de_k = \sum_{k=1}^{M} \sum_{n=1}^{npg} \omega_n \frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x} (r_n) \left| Jac \right|$$

Gauss quadrature

r	w
-0.57735026924	-1
-0.57735026924	1



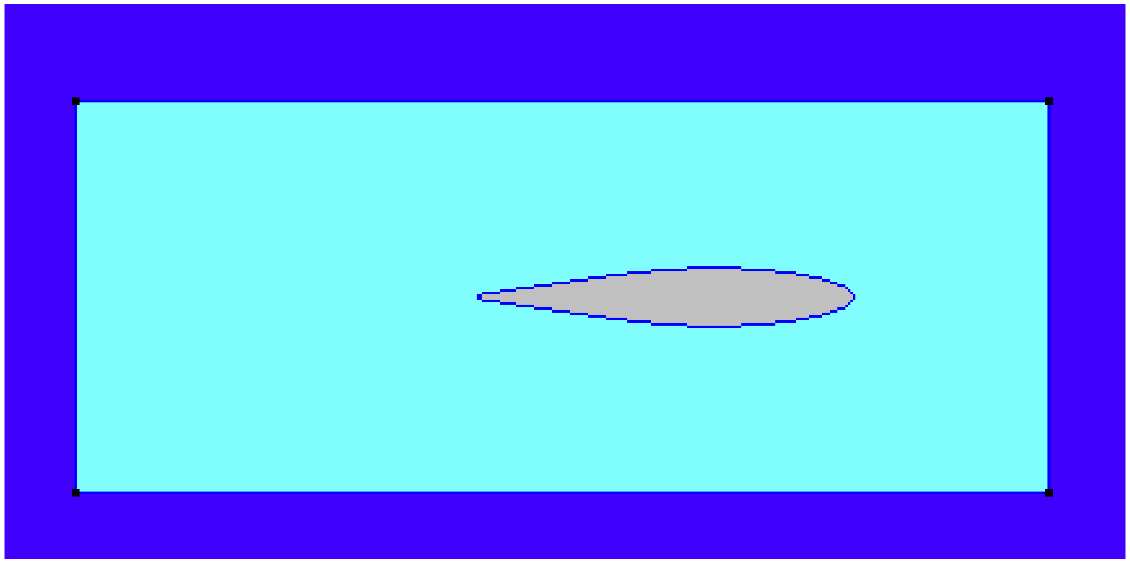
r	w
0.77459666924	0.55555555555
0	0.88888888888
-0.77459666924	0.55555555555

Errors

- Model simplifications.
- Quadrature errors.
- Linear solver errors.
- Roundoff errors.
- FEM order error.

Model simplifications

- All models have simplifications
- Example: Stationary flow around a NACA profile.



Previous errors

- Incompressible fluid→ $\nabla \cdot \bar{v} = 0$
- Non viscous and irrotational fluid (μ=0)→ $\bar{v} = \nabla \phi$
- 2D case
- Straight walls
- No gravity effects or surface tension.
- Ideal boundary conditions
- Non stationary flows

Quadrature errors

$$\int_{\Omega} f(x,y)\cdot dx\cdot dy = \sum_{i=1}^{NPG} w_i \cdot f(x_i, y_i)$$

$$(x_i, y_i) \equiv Gauss'\ points$$

$$w_i \equiv Gauss'\ coefficients$$

Errors

- Linear solver errors.

$$A \cdot x = b \rightarrow A \cdot x^{n+1} = b - r^{n+1}$$

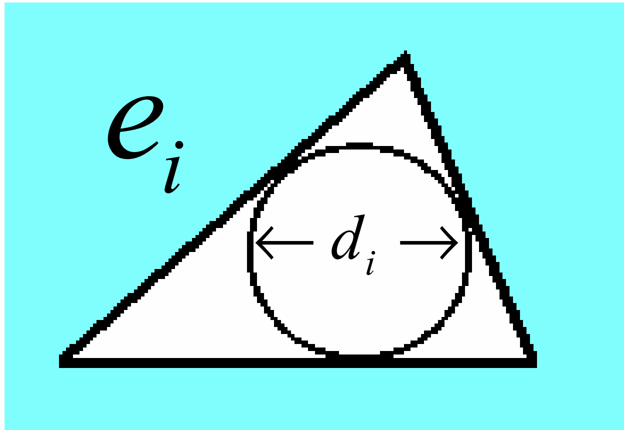
$$we \quad want \quad r^{n+1}_{n\rightarrow\infty} \rightarrow 0$$

- Truncation and roundoff (computational errors)

Error FEM

- ϕ_j order k (linear k=1)
- h = Max{d_i}

$$Error \approx h^{k+1}$$



- Under several conditions the error is limited by:

$$\sqrt{\int_\Omega |u-\tilde{u}|^2 d\Omega} \leq h^k \cdot c$$