CHAPTER III: CONICS AND QUADRICS

2. CONICS

Definition. Given a quadratic form $\omega\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$. The *projective conic* defined by ω is the set of points $X \in \mathbb{P}_2(\mathbb{R}^3)$, where ω is zero; this is,

$$\bar{C} = \{X \in \mathbb{P}_2(\mathbb{R}^3) \mid \omega(X) = 0\}.$$

And the *affine conic* defined by ω is the set of points $X \in \mathbb{A}_2$, $\tilde{X} = (1, x_1, x_2)$, where ω is zero; this is,

$$C = \{X \in \mathbb{A}_2 \mid \omega(\tilde{X}) = 0\}.$$

We have $C \subset \bar{C}$.

Using matrix notation, the equation of a conic

$$\bar{C} \equiv \sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij} x_i x_j = 0$$

can be written as follows

$$\bar{C} \equiv X^T A X = 0,$$

where

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

this is

$$X \in \bar{C} \Longleftrightarrow X^T A X = 0.$$

The equation of the projective conic is:

$$
\begin{aligned}
0 &= \sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij} x_i x_j \\
&= a_{00} x_0^2 + a_{11} x_1^2 + a_{22} x_2^2 + 2a_{01} x_0 x_1 \\
&+ 2a_{02} x_0 x_2 + 2a_{12} x_1 x_2.
\end{aligned}
$$

The equation of the affine conic is obtained substituting $x_0 = 1$:

$$
\begin{aligned}
0 = a_{00} + a_{11} x_1^2 + a_{22} x_2^2 + 2a_{01} x_1 + \\
+ 2a_{02} x_2 + 2a_{12} x_1 x_2.
\end{aligned}
$$

We say a projective conic is *degenerate* if it is reducible (its equation is a product of two polynomials of degree one), otherwise we call it *non-degenerate*.

Remember.

Definition. A *quadratic form* $\omega\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a transformation such that there exists a bilinear form $f\colon \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $\omega(v) = f(v, v)$, for every $v \in \mathbb{R}^3$.

Result. Given a quadratic form ω there exists a bilinear form f such that:

1. f is symmetric (this is, $f(u, v) = f(v, u)$),
2. the quadratic form associated to f is ω,
3. f is unique.

We call *polar form* of ω the only symmetric bilinear form of f whose quadratic form is ω.

The polar form of a quadratic form is given as follows:

$$f(u, v) = \frac{1}{2}(\omega(u + v) - \omega(u) - \omega(v)).$$

We have:

$$\omega(X) = f(X, X).$$

2.1 Singular points

Definition. Let $\bar{C}$ be a projective conic determined by a quadratic form ω, with polar form f and associated matrix A.

- We say that two points $P, Q \in \mathbb{P}_2$ are *conjugated* if $f(P,Q) = 0$.
- We say that a point $P \in \mathbb{P}_2$ is an *autoconjugated* point if $\omega(P) = f(P,P) = 0$.
- We say that a point $P \in \mathbb{P}_2$ is a *singular point* of $\bar{C}$ is it is conjugated with any point of $\mathbb{P}_2$; this is, $f(P,Q) = 0$ for every point $Q \in \mathbb{P}_2$. This is, if

$$f(P,Q) = P^T A Q = 0, \ \forall Q \in \mathbb{P}_2,$$

  or, equivalently,

$$P^T A = 0.$$

- We say that a point $P \in \mathbb{P}_2$ is a *regular point* of $\bar{C}$ if it is not a singular point.

- The conic $\bar{C}$ is *non degenerate, regular or ordinary* if it does not have a singular point.
- The conic $\bar{C}$ is *degenerate or singular* if it has a singular point.

Examples

$\bar{C}_1 \equiv x_0^2 + 2x_1^2 + 3x_1x_2 = 0$ is a non-degenerate conic, because the homogeneous polynomial of degree 2, $x_0^2 + 2x_1^2 + 3x_1x_2 = 0$ is irreducible (we cannot express it as the product of two polynomials of degree 1).

$C_2 \equiv x_0^2 - 4x_1^2 = 0$ is degenerate because $x_0^2 - 4x_1^2 = (x_0 - 2x_1)(x_0 + 2x_1)$; this is, the conic C_2 are two lines that intersect.

$\bar{C}_3 \equiv (x_0 + 2x_1 + 3x_2)^2 = 0$ is degenerate. The conic C_3 is a double line.

Obervations: Let $\bar{C}$ be a projective conic determined by a quadratic form ω, with polar form f and associated matrix A.

1. Let $Sing(\bar{C})$ be the set of singular points of $\bar{C}$, we call it singular locust of $\bar{C}$; this is,

$$\begin{aligned} Sing(\bar{C}) &= \{X \in \mathbb{P}_2 \mid f(X,Y) = 0, \text{ for every } Y \in \mathbb{P}_2\} \\ &= \{X \in \mathbb{P}_2 \mid AX = 0\}. \end{aligned}$$

   We have

$$\dim(Sing(\bar{C})) = 2 - \operatorname{rank}(A).$$

2. If $X \in \mathbb{P}_2$ is a singular point, then $X \in \bar{C}$.

   Proof. We have to prove that $\omega(X) = 0$. We have $\omega(X) = f(X,X) = 0$ as X is conjugated with any point, in particular with itself.

3. The line determined by a singular point X and any point that belongs to a conic $Y \in \bar{C}$, is contained in the mentioned conic.

Proof. As X is singular, we know that $\omega(X) = 0$ and $f(X, Y) = 0$ and as Y belongs to the conic $\omega(Y) = 0$. Any point of the line determined by X and Y has the form $Z = \lambda X + \mu Y$. We have to check that $\omega(Z) = 0$. We have:

$$\begin{aligned}
\omega(Z) &= \omega(\lambda X + \mu Y) = f(\lambda X + \mu Y, \lambda X + \mu Y) \\
&= f(\lambda X, \lambda X + \mu Y) + f(\mu Y, \lambda X + \mu Y) \\
&= f(\lambda X, \lambda X) + f(\lambda X, \mu Y) + f(\mu Y, \lambda X) + f(\mu Y, \mu Y) \\
&= \lambda^2 f(X, X) + 2\lambda\mu f(X, Y) + \mu^2 f(Y, Y) \\
&= \lambda^2 \underbrace{\omega(X)}_{0} + 2\lambda\mu \underbrace{f(X, Y)}_{0} + \mu^2 \underbrace{\omega(Y)}_{0} = 0.
\end{aligned}$$

4. All the points contained in the line joining two singular points are singular.

Proof. Let $Z = \lambda X + \mu Y$ be any point contained in a line formed by two singular points X and Y. We have to check $f(Z,T) = 0$, for every $T \in \mathbb{P}_2$. We have:

$$\begin{aligned} f(Z,T) &= f(\lambda X + \mu Y, T) \\ &= f(\lambda X, T) + f(\mu Y, T) \\ &= \lambda \underbrace{f(X,T)}_{0} + \mu \underbrace{f(Y,T)}_{0} = 0. \end{aligned}$$

5. If the conic $\bar{C}$ contains a singular point, then $\bar{C}$ is formed by lines that contain that point.

2.2 Projective classification of conics

Let $\bar{C}$ be a conic with associated matrix A.

We will say that the conic $\bar{C}$ is empty if it has no real points.

$\mathrm{rank}A$	$sign(A)$	Conic	Canon equation
3	3	Empty non-degenerate conic	$x_0^2+x_1^2+x_2^2=0$
3	1	Non empty non-degenerate conic	$x_0^2+x_1^2-x_2^2=0$
2	2	a singular point	$x_0^2+x_1^2=0$
2	0	pair of lines	$x_0^2-x_1^2=0$
1	1	double line	$(ax_0+bx_1+cx_2)^2=1$

Notation: We name *signature* of A and we denote it by $sign(A)$ to $|\alpha-\beta|$ where α is the number of positive eigenvalues of A and β is the number of negative eigenvalues of A.

2.3 Polarity defined by a conic

Let $\bar{C}$ be a conic with polar form f and associated matrix A. Let $P \in \mathbb{P}_2$, we call *polar variety* of P with respect to the conic $\bar{C}$ to the set of all conjugated points with P; this is,

$$V_P = \{X \in \mathbb{P}_2 \mid f(P, X) = 0\}.$$

If P is a singular point, then $V_P = \mathbb{P}_2$.

If P is not a singular point, then V_P is a line that we denote by r_P and call *polar line* of P with respect to the conic $\bar{C}$.

Therefore, the polar line of a non singular point $P \in \mathbb{P}_2$ is the set of points conjugated with P.

2.3.1 Equation of the polar line

If P is a non singular point with coordinates $[p_0, p_1, p_2]$ and the matrix associated to the conic is

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

then

$$r_P = \{X \in \mathbb{P}_2 \mid P^T A X = 0\},$$

this is,

$$0 = P^T A X = (p_0, p_1, p_2) \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= (p_0 a_{00} + p_1 a_{01} + p_2 a_{02}) x_0 + (p_0 a_{01} + p_1 a_{11} + p_2 a_{12}) x_1$$

$$+ (p_0 a_{02} + p_1 a_{12} + p_2 a_{22}) x_2.$$

2.3.2 Pole of a line with respect to a conic $\bar{C}$

Definition. Given a line r of the projective plane $\mathbb{P}_2$, we call *pole* of the line r with respect to the conic $\bar{C}$ the point whose polar line is r; this is, $r_P = r$.

If the equation of the line r is

$$r \equiv u_0x_0 + u_1x_1 + u_2x_2 = U^T X = 0,$$

$$\text{with } U = (u_0, u_1, u_2) \text{ and } X = (x_0, x_1, x_2),$$

then $r_P = r$ if and only if

$$P^T AX = U^T X, \text{ for every } X \in \mathbb{P}_2$$

or equivalently,

$$P^T A = U^T \Longleftrightarrow AP = U.$$

If the conic $\bar{C}$ is non-degenerate (therefore, $\det A \neq 0$), then $P = A^{-1}U$.

Theorem. If the polar line of a point Q contains a point P, then the polar line of P contains the point Q.

This is due to the conjugation condition $f(P,Q) = 0$, which is symmetric in P and Q.

2.3.3 Polarity defined by a conic

As we have seen, given a conic $\bar{C}$ every non singular point $P \in \mathbb{P}_2$ is assigned a line (its polar line) and reciprocally, every line r is assigned a point (its pole).

Definition. We call *polarity defined by a conic* $\bar{C}$ the transformation that makes every point in which is not a singular point of $\bar{C}$ correspond with its polar line, this is,

$$\begin{aligned} \mathbb{P}_2 \diagdown Sing(\bar{C}) &\longrightarrow \text{Lines of } \mathbb{P}_2 \\ P &\longmapsto r_P \end{aligned}$$

Theorem of polarity defined by a regular conic.

All the polar lines of the points of a line r of $\mathbb{P}_2$, with respect to a regular conic $\bar{C}$, contain the same point which is the pole of r.

2.4 Intersection between a line and a conic

Let $\bar{C}$ be a projective conic with polar form f and associated matrix A and let r be a projective line that contains the points $P = [p_0, p_1, p_2]$ and $Q = [q_0, q_1, q_2]$.

A point $X \in \mathbb{P}_2$ is in the intersection between the conic and the line if and only if:

$$\left\{ \begin{array}{l} X \in r \\ X \in \bar{C} \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} X = \lambda P + \mu Q \\ \omega(X) = 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} X = \lambda P + \mu Q \\ \omega(\lambda P + \mu Q) = 0 \end{array} \right.$$

The condition $\omega(\lambda P + \mu Q) = 0$ is written:

$$0 = \lambda^2 \omega(P) + 2\lambda\mu f(P, Q) + \mu^2 \omega(Q).$$

Dividing the above mentioned equation by μ^2 and writing $t = \lambda/\mu$ we obtain the following second degree equation:

$$0 = \omega(P)t^2 + 2f(P, Q)t + \omega(Q)$$

with discriminant

$$\Delta = f(P,Q)^2 - \omega(P)\omega(Q).$$

- If $f(P,Q) = 0$, $\omega(P) = 0$ and $\omega(Q) = 0$, then $P, Q \in \bar{C}$ and, therefore, $r \subset \bar{C}$. Then the conic is formed by lines.
- If not, every coefficient of the second degree equation $0 = \omega(P)t^2 + 2f(P,Q)t + \omega(Q)$ is non zero, then there are two intersection points (the two solutions of the equation).

1. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) > 0$, the line and the conic intersect in two different proper points. We say that the line is a *secant line* to the conic.
2. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) = 0$, the line and the conic intersect in a double point. We say that the line is a *tangent line* to the conic.
3. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) < 0$, the line and the conic intersect in two different points at infinity. We say that the line is an *exterior line* to the conic.

2.4.1 Tangent variety to a conic.

Definition. The *tangent variety* to a conic $\bar{C}$ at a point $P \in \bar{C}$, is the set of points $X \in \mathbb{P}_2$ such that the line that joins P and X is tangent to the conic $\bar{C}$; this is,

$$\begin{aligned} T_P\bar{C} &= \{X \in \mathbb{P}_2 \mid \Delta = f(P,X)^2 - \omega(P)\omega(X) = 0\} \\ &= \{X \in \mathbb{P}_2 \mid f(P,X) = 0\}. \end{aligned}$$

Remarks

1. If $P \in \bar{C}$ is a regular point, then $T_P\bar{C}$ is a line and, in fact, is the polar line of the point P; this is, $T_P\bar{C} = r_p$.
2. If $P \in \bar{C}$ is a singular point, then $T_P\bar{C} = \mathbb{P}_2$.
3. If $P \notin \bar{C}$, we can define the *tangent variety* to $\bar{C}$ at $P \notin \bar{C}$ as the set of points $X \in \mathbb{P}_2$ such that the line that joins P and X is tangent to the

conic $\bar{C}$; this is,

$$\begin{aligned} T_P\bar{C} &= \{X \in \mathbb{P}_2 \mid \text{line } XP \text{ is tangent to } \bar{C}\} \\ &= \{X \in \mathbb{P}_2 \mid \Delta = f(P,X)^2 - \omega(P)\omega(X) = 0\} \\ &= \{X \in \mathbb{P}_2 \mid f(P,X)^2 = \omega(P)\omega(X)\}. \end{aligned}$$

So $T_P\bar{C}$ is a degenerate conic that has P as singular point.