CHAPTER III: CONICS AND QUADRICS

3. PENCILS OF CONICS

Given the cartesian equations $r_1 = 0$ and $r_2 = 0$ of two distinct lines in the plane, the equation

$$\lambda r_1 + \mu r_2 = 0$$

is the cartesian equation of the pencil of lines that contain the unique intersection point of r_1 and r_2.

Let us consider pencils of conics instead of lines.

Definition. Given two conics $\overline{C}_1, \overline{C}_2 \subset \mathbb{P}_2$ defined by the quadratic forms

$$\omega_1 \colon \mathbb{R}^3 \longrightarrow \mathbb{R} \text{ and } \omega_2 \colon \mathbb{R}^3 \longrightarrow \mathbb{R}$$

respectively, with equations:

$$\overline{C}_1 \equiv X^t A_1 X = 0 \text{ and } \overline{C}_2 \equiv X^t A_2 X = 0,$$

we call pencil of conic $\mathcal{H}$ *determined by* $\overline{C}_1$ and $\overline{C}_2$ to the family of conics determined by the quadratic forms:

$$\lambda\omega_1 + \mu\omega_2 \colon \mathbb{R}^3 \longrightarrow \mathbb{R},\ \lambda, \mu \in \mathbb{R}.$$

If A_1 and A_2 are the matrices defining the quadratic forms w_1 and w_2 then the equation of the pencil of conics is (that is of each conic of the pencil):

$$\mathcal{H} \equiv \overline{C}_{\lambda,\mu} \equiv X^t \left(\lambda A_1 + \mu A_2\right) X = 0, \; X \in \mathbb{P}_2.$$

In particular, the conics $\overline{C}_1$ and $\overline{C}_2$, belong to this family of conics, taking parameters $\lambda = 1, \mu = 0$ and $\lambda = 0, \mu = 1$, respectively.

The intersection of the conics $\overline{C}_1$ and $\overline{C}_2$ could be a finite or infinite number of points. The intersection is infinite if both conics are the same or if they are degenerate conics with a line in common. We will exclude this two trivial cases.

We assume that $\overline{C}_1$ and $\overline{C}_2$ have a finite number of intersection points, which are the solutions of the system of quadratic equations:

$$\left\{ \begin{array}{l} X^t A_1 X = 0 \\ X^t A_2 X = 0 \end{array} \right.$$

This system has 4 solutions (not necessarily all different):

1. 4 real distinct solutions.
2. 2 real distinct solutions and 1 double solution.
3. 2 double solutions.
4. 1 solutions of order 4.
5. 2 real distinct solutions and 2 complex conjugate solutions.
6. $\ldots$

We will only study cases 1, 2, 3 and 4.

Observation.

1. The pencil of conics $\mathcal{H}$ is determined by any two conic of $\mathcal{H}$. Thus we use 2 degenerate conics, which are easier to determine.
2. Given two conics $\overline{C}_1$ and $\overline{C}_2$ in $\mathcal{H}$ then any other conic $\overline{C}$ in the pencil of conic $\mathcal{H}$ contains the intersection points of $\overline{C}_1$ and $\overline{C}_2$. That is,

$$\overline{C} \cup \overline{C}_1 = \overline{C} \cup \overline{C}_2 = \overline{C}_1 \cup \overline{C}_2$$

Definition. We call base points of a pencil of conics to the set of points in $\mathbb{P}_2$ which are in every conic of the pencil. This is, they are the intersection points of two conics that determine the pencil.

Therefore a pencil of conics $\mathcal{H}$ is determined by the base points of $\mathcal{H}$ (4 not necessarily different points).

Proposition. A pencil of conics in $\mathbb{P}_2$ contains three degenerate conics or less, unless the pencil is entirely composed by degenerate conics.

Example. The conics $\overline{C}_1, \overline{C}_2$ of equations:

$$\begin{aligned}\overline{C}_1 &\equiv x_0^2 + 2x_1^2 + x_2^2 + 4x_0x_2 = 0,\\ \overline{C}_2 &\equiv -x_1^2 + 2x_0x_2 = 0,\end{aligned}$$

give us the pencil

$$\lambda\overline{C}_1 + \mu\overline{C}_2 \equiv (x_0, x_1, x_2)\begin{pmatrix} \lambda & 0 & 2\lambda+\mu \\ 0 & 2\lambda-\mu & 0 \\ 2\lambda+\mu & 0 & \lambda \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0.$$

The degenerate conics verify:

$$0 = \begin{vmatrix} \lambda & 0 & 2\lambda+\mu \\ 0 & 2\lambda-\mu & 0 \\ 2\lambda+\mu & 0 & \lambda \end{vmatrix} = (2\lambda-\mu)\left(\lambda^2 - (2\lambda+\mu)^2\right).$$

This is, or $\mu = 2\lambda$ or

$$\lambda^2 = (2\lambda + \mu)^2 \Longleftrightarrow \begin{cases} \lambda = 2\lambda + \mu \\ \lambda = -2\lambda - \mu \end{cases} \Longleftrightarrow \begin{cases} \mu = -\lambda \\ \mu = -3\lambda \end{cases}$$

The degenerate conics of this pencil are the following conics

$$\begin{aligned} \overline{C}_1 + 2\overline{C}_2 &\equiv x_0^2 + 8x_0x_2 + x_2^2 = (x_0 + 4x_2)^2 - 15x_2^2 \\ &= (x_0 + 4x_2 + 15x_2)(x_0 + 4x_2 - 15x_2) = 0, \\ \overline{C}_1 - \overline{C}_2 &\equiv x_0^2 + 3x_1^2 + x_2^2 + 2x_0x_2 = (x_0 + x_2)^2 + 3x_1^2 = 0, \\ \overline{C}_1 - 3\overline{C}_2 &\equiv x_0^2 + 5x_1^2 + x_2^2 - 2x_0x_2 = (x_0 - x_2)^2 + 5x_1^2 = 0. \end{aligned}$$

So, $\overline{C}_1 + 2\overline{C}_2$ is the pair of lines

$$r_1 \equiv x_0 + 4x_2 + 15x_2 = 0, \text{ and } r_2 \equiv x_0 + 4x_2 - 15x_2 = 0,$$

$\overline{C}_1 - \overline{C}_2$ is the pair of lines

$$s_1 \equiv x_0 + x_2 = 0 \text{ and } s_2 \equiv x_1 = 0$$

and $\overline{C}_1 - 3\overline{C}_2$ is the pair of lines

$$l_1 \equiv x_0 - x_2 = 0 \text{ and } l_2 \equiv x_1 = 0.$$

Observation. Let us suppose that the base points A, B, C, D of a pencil of conics $\mathcal{H}$ are all different. Then $\mathcal{H}$ contains exactly three degenerate conics, which are pairs of lines:

$$\begin{aligned}
\overline{C}_1 &\equiv r(A \cup B)r(C \cup D), \\
\overline{C}_2 &\equiv r(A \cup C)r(B \cup D), \\
\overline{C}_3 &\equiv r(A \cup D)r(B \cup C)
\end{aligned}$$

Proposition. A point that does not belong to the pencil basis is contained only in one conic of the pencil.

Definition. The singular points of the degenerate conics of the pencil are called *fundamental points* of the pencil.

In general a conic $\overline{C}$ is determined by 5 different points because the matrix ρA, $\rho \neq 0$ of the conic

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

has 6 variables $a_{00}, a_{01}, a_{02}, a_{11}, a_{12}, a_{22}$ to be determined.

3.1 Some special types of pencils of conics

We will study the three types of pencils of conics which are the most commonly used in applications.

3.1.1 General pencil: four simple points

The pencil of conics $\mathcal{H}$ has four different base points A, B, C and D. Therefore, $\mathcal{H}$ has three degenerate conics:

$$\begin{aligned}
\overline{C}_1 &\equiv r(A \cup B) r(C \cup D), \\
\overline{C}_2 &\equiv r(A \cup C) r(B \cup D), \\
\overline{C}_3 &\equiv r(A \cup D) r(B \cup C).
\end{aligned}$$

Then $\mathcal{H}$ is determined by any pair of degenerate conics, that is

$$\begin{aligned}\mathcal{H} &\equiv \lambda\overline{C}_1 + \mu\overline{C}_2 = 0,\\ \mathcal{H} &\equiv \lambda\overline{C}_2 + \mu\overline{C}_3 = 0,\\ \mathcal{H} &\equiv \lambda\overline{C}_1 + \mu\overline{C}_3 = 0.\end{aligned}$$

We will use this kind of pencils to determine a conic when we have 5 points of the conic or 4 points and another condition, for example being of a kind or being tangent to a given line, etc...

Example. Determine the conic that contains the points $A(1,0,1)$, $B(1,1,1)$, and $C(1,-1,0)$ and whose center is $Z(2,1,0)$.

As the center Z is the center of the symmetry we can calculate the symmetric points of the points A and B and obtain this way two more points of the conic.

Let us take the coordinates of the point Z in the affine plane $x_0 = 1$. Therefore, we are taking the coordinates $(1, \frac{1}{2}, 0)$ of Z.

The symmetric points of A is: $A' = 2Z - A = 2(1, \frac{1}{2}, 0) - (1,0,1) = (1,1,-1)$ and the symmetric point of B is $B' = 2Z - B = 2(1, \frac{1}{2}, 0) - (1,1,1) = (1,0,-1)$.

We have four points of the conic A, A', B, B' therefore, we can determine the pencil of conics that contains these mentioned points.

Two conics of the pencil are the degenerate conics: $C_1 \equiv r(A \cup A')r(B \cup B')$ and $C_2 \equiv r(A \cup B)r(A' \cup B')$.

Let us compute the cartesian equations of these lines:

$$r(A \cup A') \equiv \begin{vmatrix} x_0 & 1 & 1 \\ x_1 & 0 & 1 \\ x_2 & 1 & -1 \end{vmatrix} = 2x_1 - x_0 + x_2 = 0,$$

$$r(B \cup B') \equiv \begin{vmatrix} x_0 & 1 & 1 \\ x_1 & 1 & 0 \\ x_2 & 1 & -1 \end{vmatrix} = 2x_1 - x_0 - x_2 = 0.$$

Then $\overline{C}_1 \equiv (2x_1 - x_0 + x_2)\,(2x_1 - x_0 - x_2) = 0$. And

$$r(A \cup B) \equiv \begin{vmatrix} x_0 & 1 & 1 \\ x_1 & 0 & 1 \\ x_2 & 1 & 1 \end{vmatrix} = x_2 - x_0 = 0,$$

$$r(A' \cup B') \equiv \begin{vmatrix} x_0 & 1 & 1 \\ x_1 & 1 & 0 \\ x_2 & -1 & -1 \end{vmatrix} = -x_0 - x_2 = 0,$$

so $\overline{C}_2 \equiv (x_2 - x_0)\,(x_0 + x_2) = 0$.

The pencil of conics $\mathcal{H}$ has equation:

$$\lambda\overline{C}_1 + \mu\overline{C}_2 \equiv \lambda\left(2x_1 - x_0 + x_2\right)\left(2x_1 - x_0 - x_2\right) + \mu\left(x_2 - x_0\right)\left(x_0 + x_2\right) = 0.$$

As it contains the point $C(1,-1,0)$ the following expression is verified:

$$\lambda\overline{C}_1 + \mu\overline{C}_2 \equiv \lambda\left(-2-1\right)\left(-2-1\right) + \mu\left(-1\right) = 0 \Longrightarrow \mu = 9\lambda.$$

The the conic we are looking for is:

$$\begin{aligned}\overline{C}_1 + 9\overline{C}_2 &\equiv \left(2x_1 - x_0 + x_2\right)\left(2x_1 - x_0 - x_2\right) + 9\left(x_2 - x_0\right)\left(x_0 + x_2\right) = 0\\ &\equiv 4x_1^2 - 8x_0^2 - 4x_0x_1 + 8x_2^2 = 0\end{aligned}$$

3.1.2 Two simple points and one double point

The conic pencil $\mathcal{H}$ has three base points A, B, C and one of them, say C, is a double point.

This case is the limit situation of the former case (when $C = D$) and it has two degenerate conics:

$$\overline{C}_1 \equiv r(A \cup B)r(C \cup C) = r_1 r_2,$$
$$\overline{C}_2 \equiv r(A \cup C)r(B \cup C) = s_1 s_2,$$

the conics $\overline{C}_2$ and $\overline{C}_3$ of the former case are the same one.
The line r_2 is tangent to any regular conic of the pencil. This pencil can be expressed as:

$$\lambda r_1 r_2 + \mu s_1 s_2 = 0.$$

This kind of pencil can be useful to determine conics when we have, for example: a line tangent to a conic and the point of tangency (this would be the line r_2), that contains three or two points and any other piece of information about the conic, for example the type, etc.

Example. Determine the conic $\overline{C}$ which is tangent to the conic $\overline{C}_1 \equiv -4x_0^2 + x_1^2 + 4x_2^2 = 0$ at the point $P(1,2,0)$, if we know that it contains the point $Q(1,0,2)$ and that $\overline{C} \cap \overline{C}_1$ are the intersection points of $\overline{C}_1$ with the line $r \equiv -3x_0 + 2x_1 + x_2 = 0$.

Therefore, the polar line of the point $P \in \overline{C}_1$ is tangent to $\overline{C}_1$ in P. Let us calculate the polar line of P:

$$t \equiv (1,2,0) \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = -4x_0 + 2x_1 = 0.$$

Let us use the following pencil of conics, this is: $\lambda\overline{C}_1 + \mu rt = 0$, this is;

$$\lambda\left(-4x_0^2 + x_1^2 + 4x_2^2\right) + \mu\left(-3x_0 + 2x_1 + x_2\right)\left(-4x_0 + 2x_1\right) = 0.$$

As $\overline{C}$ contains the point $Q(1,0,2)$ the following equation is verified:

$$12\lambda + 4\mu = 0 \Longrightarrow \mu = -3\lambda.$$

So the conic we are looking for is:

$$\overline{C} \equiv \left(-4x_0^2 + x_1^2 + 4x_2^2\right) - 3\left(-3x_0 + 2x_1 + x_2\right)\left(-4x_0 + 2x_1\right) = 0.$$

3.1.3 Two double points

The pencil has two degenerate conics: one is formed by the pair of lines $r_A r_B$ and the other is formed by the line of double points r.

The equation of the pencil is

$$\overline{C}_{\lambda,\mu} \equiv \lambda r_A r_B + \mu r^2 = 0.$$

The family of all the hyperbolas which have two lines r and s as asymptotes, form a pencil of this type with equation:

$$\overline{C}_{\lambda,\mu} \equiv \lambda r s + \mu x_0^2 = 0.$$

Example. Determine the conic $\overline{C}$ such that:

1. Contains the origin of the coordinate system and the tangent at the origin is the line $r \equiv 6x_1 + x_2 = 0$.
2. It is tangent to the circle $\overline{C}_1$ with center $(4, -6)$ and radius 2 at the point $Q(4, -4)$.
3. The axes of the conic are parallel to the coordinate axes.

The origin of the coordinate system is the point $P\,(1, 0, 0)$ and the given circle is:

$$(x_1 - 4)^2 + (x_2 + 6)^2 = 4 \Longrightarrow 12x_2 - 8x_1 + x_1^2 + x_2^2 + 48 = 0.$$

In the projective plane, the circle has equation:

$$12x_0x_2 - 8x_0x_1 + x_1^2 + x_2^2 + 48x_0^2 = 0.$$

The tangent to $\overline{C}_1$ in the point $Q\left[(1,4,-4)\right]$ is

$$r_Q \equiv (1,4,-4)\begin{pmatrix} 48 & -4 & 6 \\ -4 & 1 & 0 \\ 6 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 8x_0 + 2x_2 = 0.$$

The line that contains P and Q has equation:

$$s \equiv \det\begin{pmatrix} x_0 & 1 & 1 \\ x_1 & 0 & 4 \\ x_2 & 0 & -4 \end{pmatrix} = 4x_1 + 4x_2 = 0.$$

Let us consider the pencil of conics tangent to the line $r \equiv 6x_1 + x_2 = 0$ at the point P and which contains the point P, $C_{\lambda,\mu} \equiv \lambda r \cdot r_Q + \mu s^2 = 0$, this is,

$$\overline{C}_{\lambda,\mu} \equiv \lambda\left(6x_1 + x_2\right)\left(4x_0 + x_2\right) + \mu\left(x_1 + x_2\right)^2 = 0;$$

so, $\overline{C}_{\lambda,\mu} \equiv \lambda\left(24x_0x_1 + 4x_0x_2 + 6x_1x_2 + x_2^2\right) + \mu\left(x_1^2 + x_2^2 + 2x_1x_2\right) = 0$

The matrix of the pencil is

$$A = \begin{pmatrix} 0 & 12\lambda & 2\lambda \\ 12\lambda & \mu & 3\lambda + \mu \\ 2\lambda & 3\lambda + \mu & \lambda + \mu \end{pmatrix}$$

As the axes of the conic we are looking for are parallel to the coordinate axes the matrix

$$A_{00} = \begin{pmatrix} \mu & 3\lambda + \mu \\ 3\lambda + \mu & \lambda + \mu \end{pmatrix}$$

ought to be a diagonal matrix. Therefore, $\mu = -3\lambda$. The conic we are seeking has equation:

$$\overline{C} \equiv (6x_1 + x_2)(4x_0 + x_2) - 3(x_1 + x_2)^2 = 0.$$

3.1.4 Examples

1. **Example** Determine the conic $\overline{C}$ with asymptotes the line $2x_0 - x_1 - 2x_2 = 0$ and a line parallel to the line $x_1 = 0$ and such that the point $P(1,1,0)$ is the pole of the line $2x_1 + x_2 - 3x_0 = 0$ with respect to the conic

Solution A line parallel to the line $x_1 = 0$ has equation: $bx_0 + x_1 = 0$. Therefore, the asymptotes of $\overline{C}$ are the lines $r_1 \equiv 2x_0 - x_1 - 2x_2 = 0$ and $r_2 \equiv bx_0 + x_1 = 0$. The conic $\overline{C}$ has two improper points P_1, P_2 as it has two different asymptotes.

Therefore, the conic we are looking for is in the following pencil:

$$\overline{C}_{\lambda,\mu} \equiv \lambda r_1 \cdot r_2 + \mu x_0^2 = 0;$$

$$\text{this is, } \overline{C}_{\lambda,\mu} \equiv \lambda (2x_0 - x_1 - 2x_2)(bx_0 + x_1) + \mu x_0^2 = 0,$$

with associated matrix

$$A = \begin{pmatrix} \mu + 2b\lambda & \lambda - \frac{b}{2}\lambda & -b\lambda \\ \lambda - \frac{b}{2}\lambda & -\lambda & -\lambda \\ -b\lambda & -\lambda & 0 \end{pmatrix}$$

Last, as the point P is the pole of the line $2x_1 + x_2 - 3x_0 = 0$ with respect to $\overline{C}$, any point of the mentioned line and the point P are conjugated. Let us take the points $Q_1\,(1,0,3)$ and $Q_2\,(0,1,-2)$ of the line. As Q_1 and Q_2 are conjugated with P we have:

$$0 = (1,0,3)\begin{pmatrix} \mu + 2b\lambda & \lambda - \frac{b}{2}\lambda & -b\lambda \\ \lambda - \frac{b}{2}\lambda & -\lambda & -\lambda \\ -b\lambda & -\lambda & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -\left(2 + \frac{3}{2}b\right)\lambda + \mu,$$

$$0 = (0,1,-2)\begin{pmatrix} \mu + 2b\lambda & \lambda - \frac{b}{2}\lambda & -b\lambda \\ \lambda - \frac{b}{2}\lambda & -\lambda & -\lambda \\ -b\lambda & -\lambda & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \left(2 + \frac{3}{2}b\right)\lambda.$$

From where we obtain that $\mu = 0$ and $b = -4/3$. The conic is:

$$\overline{C} \equiv (2x_0 - x_1 - 2x_2)\left(-\frac{4}{3}x_0 + x_1\right) = 0.$$

2. Example Determine the conic $\overline{C}$ that contains the origin and has the same asymptotes as the conic $\overline{C}_1 \equiv 2x_0^2+3x_1^2-3x_2^2-2x_0x_1-14x_1x_2=0$.

Solution The conic $\overline{C}_1$ has associated matrix

$$A_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -7 \\ 0 & -7 & -3 \end{pmatrix}$$

and it is of hyperbolic type because $\det A_{00} = -9-49 < 0$. Therefore, it has two improper points P_1, P_2 that are the same as the improper points of the yet undetermined conic.

The conic $\overline{C}$ is a conic of the pencil that has equation:

$$\overline{C}_{\lambda,\mu} \equiv \lambda C_1 + \mu x_0^2 = 0;$$

this is, $\overline{C}_{\lambda,\mu} \equiv \lambda\left(2x_0^2+3x_1^2-3x_2^2-2x_0x_1-14x_1x_2\right)+\mu x_0^2 = 0.$

Besides, as it also contains the origin, $O = (1, 0, 0)$, $2\lambda + \mu = 0$ must be hold. Therefore, the conic we are looking for is:

$$\overline{C} \equiv \lambda \left(2x_0^2 + 3x_1^2 - 3x_2^2 - 2x_0x_1 - 14x_1x_2\right) - 2\lambda x_0^2 = 0;$$

this is, $\overline{C} \equiv 3x_1^2 - 3x_2^2 - 2x_0x_1 - 14x_1x_2 = 0.$

3. Example Determine the conic $\overline{C}$ tangent to the conic $\overline{C}_1 \equiv -x_0^2 + x_1^2 - 2x_2^2 - 4x_0x_2 = 0$ at the point $P(1,1,0)$; knowing also that it contains the points $Q(1,-1,0) \in C_1$ and $R(1,2,1)$.

Solution Therefore the polar line of the point $P \in \overline{C}_1$ is the tangent to $\overline{C}_1$ at P. Let us calculate the polar line of P:

$$t \equiv (1,1,0) \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = -x_0 + x_1 - 2x_2 = 0.$$

Let us consider the line that contains P and Q

$$r \equiv \det \begin{pmatrix} x_0 & 1 & 1 \\ x_1 & 1 & -1 \\ x_2 & 0 & 0 \end{pmatrix} = 0 \Longleftrightarrow -2x_2 = 0.$$

Let us use the following pencil of conics: $\lambda\overline{C}_1 + \mu r t = 0$; this is,

$$\lambda(-x_0^2 + x_1^2 - 2x_2^2 - 4x_0x_2) + \mu\left(-x_0x_2 + x_1x_2 - 2x_2^2\right) = 0.$$

As the conic we are looking for contains the point $R(1,2,1)$ the following expression is verified:

$$\lambda\,(-1+4-2-4) + \mu\,(-1+2-2) = 0 \Longrightarrow \mu = -3\lambda.$$

The conic we are looking for is

$$\overline{C} \equiv -x_0^2 + x_1^2 - 2x_2^2 + 2x_0x_2 - 3\left(-x_0x_2 + x_1x_2 - 2x_2^2\right) = 0.$$