CHAPTER III: CONICS AND QUADRICS

4. QUADRICS

Let $\mathbb{P}_3 = \mathbb{P}(\mathbb{R}^4)$ be the real projective tridimensional space.

Definition. A quadric $\overline{Q}$ in $\mathbb{P}_3$ determined by a quadratic form $\omega \colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ is the set of points of $\mathbb{P}_3$ defined by:

$$\overline{Q} = \{X \in \mathbb{P}_3 \mid \omega(X) = 0\}$$

Let $\mathcal{R} = \{O, B\}$ be a coordinate system in $\mathbb{A}_3$ and let

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}$$

be the matrix associated to the quadratic form ω then

$$\begin{aligned} \overline{Q} &= \{X \in \mathbb{P}_3 \mid X^t A X = 0\} \\ &= \left\{ [(x_0, x_1, x_2, x_3)] \in \mathbb{P}_3 \mid \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x_i x_j = 0 \right\} \end{aligned}$$

The affine quadric defined by the quadratic form ω is the subset Q of $\mathbb{A}_3$ defined by

$$Q = \{X \in \mathbb{A}_3 \mid \omega(\tilde{X}) = 0\},$$

where $\tilde{X} = (1, x_1, x_2, x_3)$, with $(x_1, x_2, x_3) \in \mathbb{A}_3$. It is verified that $Q \subset \overline{Q}$.

4.1 Singular points and projective classification

Let $\overline{Q}$ be a projective quadric determined by a quadratic form $\omega\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$, with polar form $f\colon \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ and associated matrix A with respect to certain coordinate system.

Definitions.

- We say that two points $A, B \in \mathbb{P}_3$ are *conjugated* with respect to $\overline{Q}$ if $f(A, B) = 0$.
- We say that a point $P \in \mathbb{P}_3$ is an *autoconjugated point* with respect to $\overline{Q}$ if $\omega(P) = f(P, P) = 0$.
- We say that a point $P \in \mathbb{P}_3$ is a *singular point* of $\overline{Q}$ if it is conjugated with every point of $\mathbb{P}_3$; this is, $f(P, X) = 0$ for every point $X \in \mathbb{P}_3$. This is, if

$$f(P, X) = P^T AX = 0, \ \forall X \in \mathbb{P}_3,$$

or equivalently,

$$P^T A = 0.$$

- We say that a point $P \in \mathbb{P}_3$ is a *regular point* of $\overline{Q}$ if it is not a singular point
- The quadric $\overline{Q}$ is *non degenerate, regular or ordinary* if it does not have singular points.
- The quadric $\overline{Q}$ is *degenerate or singular* if it has a singular point.

Observations: Let $\overline{Q}$ be a projective quadric generated by a quadratic form ω, with polar form f and associated matrix A.

1. Let $\mathrm{sign}(\overline{Q})$ be the set of singular points of $\overline{Q}$; this is,

$$\begin{aligned}\mathrm{sign}(\overline{Q}) &= \{X \in \mathbb{P}_3 \mid f(X,Y) = 0, \text{ for every } Y \in \mathbb{P}_3\} \\ &= \{X \in \mathbb{P}_3 \mid AX = 0\}.\end{aligned}$$

   We have

$$\dim(\mathrm{sign}(\overline{Q})) = 3 - \mathrm{rank}(A).$$

2. If $X \in \mathbb{P}_3$ is a singular point, then $X \in \overline{Q}$.

   Proof. We have to check that $\omega(X) = 0$. We have $\omega(X) = f(X,X) = 0$ as X is conjugated with any point, in particular with itself.

3. The line determined by a singular point X and any other point of the quadric, $Y \in \overline{Q}$, is contained on the quadric.

   Proof. As X is singular we know that $\omega(X) = 0$ and $f(X, Y) = 0$ and as Y belongs to the quadric $\omega(Y) = 0$. Any point of the line determined by X and Y has the form $Z = \lambda X + \mu Y$. We have to check whether $\omega(Z) = 0$. We have:

$$\begin{aligned}
\omega(Z) &= \omega(\lambda X + \mu Y) = f(\lambda X + \mu Y, \lambda X + \mu Y) \\
&= f(\lambda X, \lambda X + \mu Y) + f(\mu Y, \lambda X + \mu Y) \\
&= f(\lambda X, \lambda X) + f(\lambda X, \mu Y) + f(\mu Y, \lambda X) + f(\mu Y, \mu Y) \\
&= \lambda^2 f(X, X) + 2\lambda\mu f(X, Y) + \mu^2 f(Y, Y) \\
&= \lambda^2 \underbrace{\omega(X)}_{0} + 2\lambda\mu \underbrace{f(X, Y)}_{0} + \mu^2 \underbrace{\omega(Y)}_{0} = 0.
\end{aligned}$$

4. All the points that belong to the line determined by two singular points are singular.

Proof. Let $Z = \lambda X + \mu Y$ be any point of the line formed by two singular points X and Y. We have to check that $f(Z,T) = 0$, for every $T \in \mathbb{P}_3$. We have:

$$\begin{aligned} f(Z,T) &= f(\lambda X + \mu Y, T) \\ &= f(\lambda X, T) + f(\mu Y, T) \\ &= \lambda \underbrace{f(X,T)}_{0} + \mu \underbrace{f(Y,T)}_{0} = 0. \end{aligned}$$

5. If the quadric $\overline{Q}$ contains a singular point, then $\overline{Q}$ is formed by lines that contain that point.

4.1.1 Projective classification

1. If $\det A \neq 0$, then the quadric $\overline{Q}$ is *ordinary or not degenerate*.
2. If $\det A = 0$, then the quadric $\overline{Q}$ is *degenerate*.

   a) If $\operatorname{rank}(A) = 3$, then $\overline{Q}$ has an unique singular point P.
   - If P is a proper point, then $\overline{Q}$ is a *cone* with vertex P.
   - If P is an improper point, then $\overline{Q}$ is a *cylinder*.

   b) If $\operatorname{rank}(A) = 2$, then $\overline{Q}$ has a line of singular points and $\overline{Q}$ is a *pair of planes* with intersection the line of singular points.

   c) If $\operatorname{rank}(A) = 1$, then $\overline{Q}$ has a plane of singular points and $\overline{Q}$ is a *double plane*.

4.2 Polarity defined by a quadric

Let $\overline{Q}$ be a quadric with polar form f and associated matrix A. Let us consider $P \in \mathbb{P}_3$, we call *polar variety* of P with respect to the quadric $\overline{Q}$ to the set of conjugated points of P; this is,

$$\begin{aligned} V_P &= \{X \in \mathbb{P}_3 \mid f(P, X) = 0\} \\ &= \{X \in \mathbb{P}_3 \mid P^t A X = 0\}. \end{aligned}$$

If $P \in \mathbb{P}_3$ is a singular points, then $V_P = \mathbb{P}_3$.

If $P \in \mathbb{P}_3$ is not a singular point, then V_P is a plane π_P and we call it *polar plane* of P with respect to the quadric $\overline{Q}$:

$$\pi_P = \{X \in \mathbb{P}_3 \mid P^t A X = 0\}.$$

Definition. Given a plane π of the space $\mathbb{P}_3$, we call *pole* of the plane π with respect to the quadric $\overline{Q}$ to the point whose polar plane is π; this is, $\pi_P = \pi$.

If the equation of the plane π is

$$\pi \equiv u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = U^T X = 0,$$
$$\text{with } U = (u_0, u_1, u_2, u_3) \text{ and } X = (x_0, x_1, x_2, x_3),$$

then $\pi_P = \pi$ if and only if

$$P^T AX = U^T X, \text{ for every } X \in \mathbb{P}_3$$

equivalently,

$$P^T A = U^T \iff AP = U.$$

And if the quadric $\overline{Q}$ is not degenerate (therefore, $\det A \neq 0$), then $P = A^{-1}U$.

Theorem. If the point P belongs to the polar plane of a point R, then the point R is in the polar plane of P.

This is due to the condition of conjugation $f(P,R)=0$; it is symmetric in P and R.

As we have seen, given a quadric $\overline{Q}$, every non sigular point P is assigned a plane (its polar plane) and reciprocally, each plane π is assigned a point (its pole).

Definition. We call *polarity defined by a quadric* $\overline{Q}$ to the transformation in which each non singular point of $\overline{Q}$ is assigned to its polar plane. This is,

$$\begin{aligned} \mathbb{P}_3 \diagdown \operatorname{sign}(\overline{Q}) &\longrightarrow \text{Planes of } \mathbb{P}_3 \\ P &\longmapsto \pi_P \end{aligned}$$

Fundamental theorem of polarity

The polar planes of the points of a plane π of $\mathbb{P}_3$, with respect to a regular quadric $\overline{Q}$, contain the same point which is precisely the pole of π.

4.3 Intersection between line and quadric

Let $\overline{Q}$ be a projective quadric with polar form f and associated matrix A. Let r be the projective line which contains the independent points $P = [(p_0, p_1, p_2, p_3)]$ and $Q = [(q_0, q_1, q_2, q_3)]$.

A point $X \in \mathbb{P}_3$ is in the intersection between the conic and the line if and only if:

$$\left\{ \begin{array}{l} X \in r \\ X \in \overline{Q} \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} X = \lambda P + \mu Q \\ \omega(X) = 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} X = \lambda P + \mu Q \\ \omega(\lambda P + \mu Q) = 0 \end{array} \right.$$

The condition $\omega(\lambda P + \mu Q) = 0$ is written:

$$0 = \lambda^2 \omega(P) + 2\lambda\mu f(P, Q) + \mu^2 \omega(Q).$$

Dividing the former equation by μ^2 and writing $t = \lambda/\mu$ we obtain the following second degree equation:

$$0 = \omega(P)t^2 + 2f(P, Q)t + \omega(Q)$$

with discriminant

$$\Delta = f(P,Q)^2 - \omega(P)\omega(Q).$$

- If $f(P,Q) = 0$, $\omega(P) = 0$ and $\omega(Q) = 0$, then $P, Q \in \overline{Q}$ and, therefore, $r \subset \overline{Q}$.
- If not all the coefficients of the second degree equation $0 = \omega(P)t^2 + 2f(P,Q)t + \omega(Q)$ are zero, then there are two intersection points (the two solutions of the equation).

1. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) > 0$, the line, and the quadric intersect in two different real points. The line is called *secant line* to the quadric.
2. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) = 0$, the line and the quadric intersect in a double point. The line is called *tangent line* to the quadric.
3. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) < 0$, the line and the conic intersect in two different improper points. The line is called *exterior line* to the quadric.

4.3.1 Tangent variety to a quadric

Definition. The *tangent variety* to a quadric $\overline{Q}$ in a point $P \in \mathbb{P}_3$, is the set of points $X \in \mathbb{P}_3$ such that the line that joins P and X is tangent to the quadric $\overline{Q}$; this is,

$$\begin{aligned} T_P\overline{Q} &= \{X \in \mathbb{P}_3 \mid \text{line } XP \text{ is tangent to } \overline{Q}\} \\ &= \{X \in \mathbb{P}_3 \mid \Delta = f(P,X)^2 - \omega(P)\omega(X) = 0\} \\ &= \{X \in \mathbb{P}_3 \mid f(P,X)^2 = \omega(P)\omega(X)\}. \end{aligned}$$

Observations.

1. $T_P\overline{Q}$ is a degenerate quadric which has P as singular point.
2. If $P \in \overline{Q}$ is a regular point, then

$$\begin{aligned} T_P\overline{Q} &= \{X \in \mathbb{P}_3 \mid f(P,X)^2 = 0\} \\ &= \{X \in \mathbb{P}_3 \mid P^t A X = 0\} \end{aligned}$$

   is a plane, called *the tangent plane to* $\overline{Q}$ *in* P. In fact, it is the polar plane of the point P; this is, $T_P\overline{Q} = \pi_p$.

3. If $P \in \overline{Q}$ is a singular point, then $T_P\overline{Q} = \mathbb{P}_3$.

4.4 Affine classification and notable elements of quadrics

Let $\overline{\mathbb{A}}_3 = \mathbb{P}(\mathbb{R}^4)$ be the projectivized affine space, with coordinate system $\mathcal{R} = \{O, B\}$. And let ω be a quadratic form with associated matrix A. Let

$$\overline{Q} = \{X \in \mathbb{P}_3(\mathbb{R}^4) \mid \omega(X) = 0$$

be a projective quadric with affine quadric

$$Q = \overline{Q} \cap \mathbb{A}_3 = \{X \in \mathbb{A}_3 \mid \omega(\tilde{X}) = 0\}, \text{ where } \tilde{X} = (1, x_1, x_2, x_3).$$

4.4.1 Center of an affine quadric

Definition. We call *center* of an affine quadric Q to the pole of the plane at infinity, if it exists. If that point is contained in the plane at infinity then the quadric has an improper center, otherwise a proper center.

The equation of the plane at infinity is $x_0 = 0$ and the equation of the quadric is $X^tAX = 0$. Therefore, the pole of the plane at infinity is the point P such that $P^tA = (1, 0, 0, 0)$.

Proposition. The proper center of an affine quadric is the center of symmetry. Any line that contains the center of a quadric intersects the quadric in two symmetric points with respect to the center.

4.4.2 Relative position of the quadric and the plane at infinity

Let $\pi_\infty \equiv x_0 = 0$ be the equation of the plane at infinity and let us consider the projective quadric $\overline{Q}$ determined by a quadratic form ω and with associated matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

We have:

$$\overline{Q} \cap \pi_\infty = \{X \in \pi_\infty \mid \omega(X) = 0\} = \big\{(0, x_1, x_2, x_3) \mid X^t A X = 0\big\}$$

this is,

$$\overline{Q} \cap \pi_\infty \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0,$$

then $\overline{Q} \cap \pi_\infty$ is a conic of the plane at infinity π_∞ with matrix

$$A_{00} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Proposition. The quadric $\overline{Q}$ has a center if and only if $\det A_{00} \neq 0$. Besides,

- If $\det A_{00} \neq 0$, then the conic $\overline{Q} \cap \pi_\infty$ is regular and $\overline{Q}$ has a center.
- If $\det A_{00} = 0$, then the conic $\overline{Q} \cap \pi_\infty$ is degenerate and $\overline{Q}$ has no center.

4.4.3 Diameters of a quadric

Definition. We call *diameter* of a quadric $\overline{Q}$ to every line that contains the center of $\overline{Q}$.

Definition. We call *diametral plane* of a quadric $\overline{Q}$ to the planes that contain the center of $\overline{Q}$.

Definition. Two diameters D and D' are said *conjugated* if their improper points are conjugated.

Definition. We call *diametral polar plane of a diameter* D to the polar plane of the improper point of D.

4.4.4 Axes of a quadric with proper center

Definition. We call *axis* of a quadric $\overline{Q}$ to the diameter which is perpendicular to its diametral polar plane.

Let $\overline{Q}$ be a projective quadric with associated matrix A. And let A_{00} be the matrix of the conic $\overline{Q} \cap \pi_\infty$.

As $\overline{Q}$ has a proper center Z, the matrix A_{00} is not singular, so its three eigenvalues are not zero λ_1, λ_2 and λ_3.

Let v_1, v_2 and v_3 be the eigenvectors associated to λ_1, λ_2 and λ_3 respectively (we choose the eigenvectors which are orthogonal two by two).

The axes of $\overline{Q}$ are the lines that contain the center, Z, and have as directions the vectors v_1, v_2 and v_3, respectively.

The coordinate system $\mathcal{R} = \{Z, \{v_1, v_2, v_3\}\}$ gives us a cartesian autoconjugated coordinate system.

We can find three situations:

1. The three eigenvalues are different. Then $\overline{Q}$ has three axes which are orthogonal two by two.
2. An eigenvalue is double, $\lambda_1 = \lambda_2$, and the other, λ_3, is simple. Then the dimension of the subspace of eigenvectors associated to the double eigenvalue is $\dim V_1 = 2$ and $\dim V_3 = 1$. Then V_1 is a plane of axes perpendicular to the axis V_3. In this case the quadric $\overline{Q}$ is a revolution quadric, whose axis is the one that corresponds to the eigenvalue λ_3.
3. The three eigenvalues are the same, $\lambda_1 = \lambda_2 = \lambda_3$. Then any diameter is the axis and the quadric is a sphere.

Definition. We call *main planes* of a quadric $\overline{Q}$ to the diametral polar planes of the axes.

4.4.5 Asymptotic cones

Definition. We call *asymptotes* of a quadric $\overline{Q}$ to the tangents of a conic in its improper points.

Let $\overline{Q}$ be a projective quadric with proper center Z.

Definition. The tangent variety to the quadric $\overline{Q}$ from the center $Z\,[(z_0, z_1, z_2, z_3)]$ is a cone that is called *asymptotic cone*. The equation of the asymptotic cone is the following one:

$$f(Z,X)^2 - \omega(Z)\omega(X) = 0 \Longleftrightarrow (Z^tAX)(Z^tAX) - (Z^tAZ)(X^tAX) = 0$$
$$\Longleftrightarrow x_0^2 - z_0(X^tAX) = 0$$
$$\Longleftrightarrow x_0^2 - \frac{\det A_{00}}{\det A}(X^tAX) = 0$$

equivalently

$$\frac{\det A}{\det A_{00}} x_0^2 - \overline{Q} = 0.$$

The quadrics of ellyptic type have an imaginary asymptotic cone and the quadrics of hyperbolic type have a real asymptotic cone.

The *generatrixs* of the cone (lines of the cone) are the diameters tangent to the quadric.

We call *asymptotic plane* to the polar planes of the points of the improper conic of $\overline{Q}$ ($\overline{Q} \cap \pi_\infty = C$) (if there exists any).

Example 1. Let us consider the quadric $Q \equiv x_1^2 + 3x_3^2 + 4x_1x_2 + 2x_3 + 2 = 0$. The matrix of Q is:

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

The determinant of A is $\det A = -20$, quadric with proper center:

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \rho \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

this is

$$\begin{cases} 2z_0 + z_3 = \rho \\ z_1 + 2z_2 = 0 \\ 2z_1 = 0 \\ z_0 + 3z_3 = 0 \end{cases}$$

The center is: $Z\left[(1, 0, 0, -1/3)\right]$.

Equation of the asymptotic cone:

$$\frac{\det A}{\det A_{00}} x_0^2 - \overline{Q} = 0 \Longleftrightarrow \frac{20}{12} x_0^2 - (x_1^2 + 3x_3^2 + 4x_1x_2 + 2x_3x_0 + 2x_0^2) = 0$$

$$\Longleftrightarrow 2x_0x_3 + 4x_1x_2 + \frac{1}{3}x_0^2 + x_1^2 + 3x_3^2 = 0$$

4.5 Metric invariants of a quadric $\overline{Q}$

Let us consider the quadric $\overline{Q}$ with associated matrix A; this is, $\overline{Q} \equiv X^T A X = 0$. The following values are euclidean invariants of the quadric:

- $\det A$
- Eigenvalues of A_{00}: $\lambda_1, \lambda_2, \lambda_3$ or equivalently:

$$\det A_{00}, \ \operatorname{tr} A_{00} = a_{11} + a_{22} + a_{33}, \ J = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$$

where

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix} \text{ and } A_{00} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

The following identities are satisfied:

- $\det A_{00} = \lambda_1\lambda_2\lambda_3$
- $J = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$
- $\operatorname{tr} A_{00} = \lambda_1 + \lambda_2 + \lambda_3$

The charasteristic equation of A_{00} is:

$$|A_{00} - \lambda I_3| = -\lambda^3 + \operatorname{tr} A_{00}\lambda^2 - J\lambda + \det A_{00} = 0.$$

Therefore, λ_1, λ_2 and λ_3 are the roots of the equation $|A_{00} - \lambda I_3| = 0$.

If $\det A_{00} \neq 0$, then the conic $\overline{Q} \cap \pi_\infty$ is regular and $\overline{Q}$ has a center.

If $\det A_{00} = 0$, then the conic $\overline{Q} \cap \pi_\infty$ is not regular. It is a quadric of paraboloid type, it may not have a center, have a line of centers or even have a plane of centers.

4.5.1 Classification of quadrics with $\det A_{00} \neq 0$.

Because of $\det A_{00} = \lambda_1\lambda_2\lambda_3 \neq 0$, in certain coordinate system, the matrix of the quadric is

$$\begin{pmatrix} d_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

and, therefore, the reduced equation of the affine quadric ($x_0 = 0$) is

$$d_0 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

with $d_0 = \frac{\det A}{\det A_{00}}$ and they are quadrics *with center*.

If $\det A = d_0\lambda_1\lambda_2\lambda_3 \neq 0$ (this is, $\operatorname{rank}(A) = 4$) then they are *ordinary* quadrics.

We can distinguish two cases:

1. the eigenvalues of A_{00} have the same sign
2. two of the eigenvalues of A_{00} have the same sign and the other the opposite sign.

1. If $\operatorname{sign}(\lambda_1) = \operatorname{sign}(\lambda_2) = \operatorname{sign}(\lambda_3)$ $(+ + +$ o $- - -)$, we say that A_{00} has *signature* 3, $\operatorname{sig} A_{00} = 3$, and we can encounter the following cases:

   a) If $\operatorname{sign}(d_0) = \operatorname{sign}(\lambda_1) = \operatorname{sign}(\lambda_2) = \operatorname{sign}(\lambda_3)$, then $\det A > 0$ and the reduced equation of the affine quadric is

$$1 = -\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}$$

   where $a^2 = d_0/\lambda_1$, $b^2 = d_0/\lambda_2$ and $c^2 = d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *imaginary ellipsoid*.

   b) If $\operatorname{sign}(d_0) \neq \operatorname{sign}(\lambda_1) = \operatorname{sign}(\lambda_2) = \operatorname{sign}(\lambda_3)$, then $\det A < 0$ and the reduced equation of the affine quadric is

$$1 = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}$$

   where $a^2 = -d_0/\lambda_1$, $b^2 = -d_0/\lambda_2$ and $c^2 = -d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *ellipsoid*, and if besides $a^2 = b^2 = c^2$ we obtain a *sphere*.

2. If $\text{sign}(\lambda_1) = \text{sign}(\lambda_2) \neq \text{sign}(\lambda_3)$ ($+ + -$ o $- - +$) we say that A_{00} has *signature* 1, $\text{sig}\, A_{00} = 1$, and we can encounter the following cases:

   a) If $\text{sign}(d_0) \neq \text{sign}(\lambda_1) = \text{sign}(\lambda_2) \neq \text{sign}(\lambda_3)$, then $\det A > 0$ and the reduced equation of the affine quadric is

$$1 = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}$$

   where $a^2 = -d_0/\lambda_1$, $b^2 = -d_0/\lambda_2$ and $c^2 = d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *hyperbolic hyperboloid*.

   b) If $\text{sign}(d_0) = \text{sign}(\lambda_1) = \text{sign}(\lambda_2) \neq \text{sign}(\lambda_3)$, then $\det A < 0$ and the reduced equation of the quadric is

$$1 = -\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}$$

   where $a^2 = d_0/\lambda_1$, $b^2 = d_0/\lambda_2$ and $c^2 = -d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *elliptic hyperboloid*.

If $\det A = d_0\lambda_1\lambda_2\lambda_3 = 0$ (this is, $d_0 = 0$ and $\text{rank}(A) = 3$) then they are *degenerate* quadrics with reduced equation:

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

We can distinguish two cases:

1. If $\text{sign}(\lambda_1) = \text{sign}(\lambda_2) = \text{sign}(\lambda_3)$, the reduced equation of the affine quadric is

$$0 = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$$

   which is the equation of an *imaginary cone*.

2. If $\text{sign}(\lambda_1) = \text{sign}(\lambda_2) \neq \text{sign}(\lambda_3)$, the reduced equation of the affine quadric is of the form

$$0 = a^2 x_1^2 + b^2 x_2^2 - c^2 x_3^2$$

   which is the equation of an *cone*.

Table of classification of quadrics with center

$\det A_{00} \neq 0$

- $\text{rank}(A) = 4$ — Regular
  - $\text{sig}\, A_{00} = 3$ — Ellipsoids
    - $\det A > 0$ imaginary
    - $\det A < 0$ real
  - $\text{sig}\, A_{00} = 1$ — Hyperboloids
    - $\det A > 0$ hyperbolic
    - $\det A < 0$ elliptic
- $\text{rank}(A) = 3$ — Cones
  - $\text{sig}\, A_{00} = 3$ Imaginary cone with a real point
  - $\text{sig}\, A_{00} = 1$ Real cone

4.5.2 Classification of the quadrics with $\det A_{00} = 0$.

Because of $\det A_{00} = \lambda_1\lambda_2\lambda_3 = 0$, we can suppose $\lambda_3 = 0$.
Hence $J = \lambda_1\lambda_2$.
In certain coordinate system the matrix of the quadric is

$$\begin{pmatrix} b_{00} & 0 & 0 & b_{03} \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ b_{03} & 0 & 0 & 0 \end{pmatrix}$$

with $\det A = -b_{03}^2\lambda_1\lambda_2$.
The reduced equation of the affine quadric is

$$0 = b_{00} + \lambda_1 x_1^2 + \lambda_2 x_2^2.$$

If $J = \lambda_1\lambda_2 \neq 0$ we can distinguish various cases:

1. If $\det A \neq 0$ (this is, $b_{03} \neq 0$) The reduced equation of the affine quadric is

$$0 = 2b_{03}x_3 + \lambda_1 x_1^2 + \lambda_2 x_2^2$$

and we have:

a) If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2)$, this is $J > 0$, the reduced equation of the affine quadric is of the form

$$0 = dx_3 + a^2x_1^2 + b^2x_2^2$$

which is the equation of an *elliptic paraboloid*.

b) If $\mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$, this is $J < 0$, the reduced equation of the affine quadric is of the form

$$0 = dx_3 + a^2x_1^2 - b^2x_2^2$$

which is the equation of an *hyperbolic paraboloid*.

2. If $\det A = 0$ (this is, $b_{03} = 0$) the reduced equation of the affine quadric is

$$0 = b_{00} + \lambda_1 x_1^2 + \lambda_2 x_2^2$$

and we distinguish various cases:

a) If $b_{00} \neq 0$ we have

1) If $\operatorname{sign}(\lambda_1) = \operatorname{sign}(\lambda_2)$, this is $J > 0$, the reduced equation of the affine quadric is of the form

$$0 = c + a^2 x_1^2 + b^2 x_2^2$$

which is the equation of an *elliptic imaginary cylinder* if $c > 0$ or *elliptic cylinder* if $c < 0$.

2) If $\operatorname{sign}(\lambda_1) \neq \operatorname{sign}(\lambda_2)$, this is $J < 0$, the reduced equation of the affine quadric is of the form

$$0 = c + a^2 x_1^2 - b^2 x_2^2$$

which is the equation of a *hyperbolic cylinder*.

b) If $b_{00} = 0$ the reduced equation of the quadric is

$$0 = \lambda_1 x_1^2 + \lambda_2 x_2^2.$$

1) If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2)$, this is $J > 0$, the affine quadric is a *pair of imaginary planes which intersect in a line*.
2) If $\mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$, this is $J < 0$, the affine quadric is a *pair of planes which intersect in a line*.

If two of the eigenvalues of A_{00} vanish (suppose $\lambda_2 = \lambda_3 = 0$), hence: $\det A = 0$, $\det A_{00} = 0$, $J = 0$ and $\operatorname{tr} A_{00} = \lambda_1$.

In certain coordinate system the matrix of the quadric is

$$\begin{pmatrix} b_{00} & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the reduced equation of the quadric is

$$0 = b_{00} + \lambda_1 x_1^2.$$

1. If $b_{00} \neq 0$ we have

   a) If $\operatorname{sign}(b_{00}) = \operatorname{sign}(\lambda_1)$, the reduced equation of the affine quadric is of the form

   $$0 = p^2 + a^2 x_1^2$$

   and the affine quadric is a *pair of imaginary parallel planes* .

b) If $\mathrm{sign}(b_{00}) \neq \mathrm{sign}(\lambda_1)$, the reduced equation is of the form

$$0 = p^2 - a^2x_1^2 = (p + ax_1)(p - ax_1)$$

and the affine quadric is a *pair of parallel planes*.

Table of classification of quadrics with $\det A_{00} = 0$

$\det A_{00} = 0$

- $\operatorname{rank}(A) = 4$ Regular
  - $J > 0$ Elliptic paraboloid
  - $J < 0$ Hyperbolic paraboloid
- $\operatorname{rank}(A) = 3$ Cylinders
  - $J > 0$ Real elliptic cylinder
  - $J < 0$ Hyperbolic cylinder
  - $J = 0$ Parabolic cylinder
- $\operatorname{rank}(A) = 2$ Pair of planes
  - $J > 0$ Pair of imaginary planes (line)
  - $J < 0$ Pair of secant planes
  - $J = 0$ Pair of parallel planes
    - imaginary
    - real
- $\operatorname{rank}(A) = 1$ double plane