MAPS BETWEEN SETS

Let A, B and C be nonempty sets.

Definition A map $f : A \to B$ is a correspondence assigning to each element of A a unique element in B. The set A is called domain of the map f and the set B is the range of f.

Given $a \in A$ we denote by $f(a)$ the image element of a in B through the map f. The image of f is the set

$$\mathrm{Im}(f) = f(A) = \{f(a) \mid a \in A\}.$$

The inverse image of an element $b \in B$ is the set

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$

The identity map A is the map $id_A : A \to A$ defined by $id_A(a) = a$ for every $a \in A$.

Definition Given a map $f : A \to B$ we call f:

1. **Injective or one-to-one** if given two different elements of A their images through f are distinct. That is
$$\forall a_1, a_2 \in A, a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2).$$
Equivalently,
$$\text{if } \exists a_1, a_2 \in A \text{ such that } f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

2. **Surjective** if every element in B is the image through f of an element of A. That is, $f(A) = B$,
$$\forall b \in B, \exists a \in A \text{ such that } f(a) = b.$$

3. **Bijective** if it is injective and surjective.

Definition Given two maps $f : A \to B$ and $g : B \to C$, the **composition** of f with g is the map $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for every $a \in A$.

Definition Given a bijective map $f : A \to B$ the **inverse** map of f is the map $f^{-1} : B \to A$ defined by:

$$\forall b \in B, f^{-1}(b) = a \text{ if } f(a) = b \text{ with } a \in A.$$

It holds that $f^{-1} \circ f = id_A$ and that $f \circ f^{-1} = id_B$.

7. LINEAR TRANSFORMATIONS

Let V and W be real vector spaces.

Definition A map $f : V \to W$ is a linear transformation (or homomorphism) if it verifies

$$\begin{aligned} &\forall u, v \in V,\ f(u+v) = f(u) + f(v), \\ &\forall \lambda \in \mathbb{R},\ f(\lambda u) = \lambda f(u). \end{aligned}$$

This is equivalent to

$$\forall u, v \in V,\ \forall \lambda, \mu \in \mathbb{R},\ f(\lambda u + \mu v) = \lambda f(u) + \mu f(v).$$

A linear transformation $f : V \to W$ where $W = V$ is called endomorphism.

Example The map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$f(x, y, z) = (2x - y + 4z, 3x - z, 6x + y)$$

is linear.

It is also linear every transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$f(x_1, \ldots, x_n) = (\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \ldots, \sum_{i=1}^{n} l_i x_i),$$

with $a_i, b_i, \ldots, l_i \in \mathbb{R}$, $i = 1, \ldots, n$.

Proposition Let $f : V \to W$ be a linear transformation. Given a set $\{u_1, \ldots, u_m\}$ of vectors in V, the following statements hold:

1. Given $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ then

$$f(\sum_{i=1}^{m} \lambda_i u_i) = \sum_{i=1}^{m} \lambda_i f(u_i).$$

   In particular, $f(0_V) = 0_W$ and $f(-u) = -f(u)$, $\forall u \in V$.

2. If $u_1, \ldots, u_m$ are linearly dependent then

$$f(u_1), \ldots, f(u_m)$$

   are also linearly dependent.

3. If $f(u_1), \ldots, f(u_m)$ are linearly independent then $u_1, \ldots, u_m$ are linearly independent. The converse is not true, in general, the linear independence of vectors is not preserved by linear transformations.

4. If U is a vector subspace of V with basis $\{u_1, \ldots, u_m\}$ then $f(U) = \langle f(u_1), \ldots, f(u_m) \rangle$. This is $\{f(u_1), \ldots, f(u_m)\}$ is a generating set of $f(U)$ but it may not be a basis of $f(U)$.

Examples Let us consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (x, y).$$

1. The vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$ are linearly independent and so are their images, $f(v_1) = (1, 0)$ and $f(v_2) = (0, 1)$. On the other hand, the vectors $u_1 = (1, 0, 1)$ and $u_2 = (2, 0, 0)$ are linearly independent but their images $f(u_1) = (1, 0)$ and $f(u_2) = (2, 0)$ are linearly dependent.

2. Given the subspace $U = \langle u_1 = (1, 0, 1), u_2 = (2, 0, 0)\rangle$, with $\dim U = 2$, of $\mathbb{R}^3$ then $f(U) = \langle f(u_1), f(u_2)\rangle = \langle (1, 0)\rangle$ and so $\dim f(U) = 1$.

THE MATRIX OF A LINEAR TRANSFORMATION

Proposition Let $B = \{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_n\}$ a set of vectors in W. There exists a unique linear transformation $f : V \to W$ such that

$$f(v_1) = w_1, \ldots, f(v_n) = w_n.$$

Example Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear application verifying:

$$f(1,0,0) = (3,1,2,4), f(0,1,0) = (1,-1,-5,5),$$
$$f(0,0,1) = (2,-2,-3,4).$$

By the previous proposition, there exists a unique linear transformation verifying the previous conditions. Such transformation is defined by:

$$\begin{aligned} f(x,y,z) &= xf(1,0,0) + yf(0,1,0) + zf(0,0,1) = \\ &= (3x + y + 2z, x - y - 2z, 2x - 5y - 3z, 4x + 5y + 4z). \end{aligned}$$

Definition Let $B_V = \{v_1, \ldots, v_n\}$ be a basis of V and let $(x_1, \ldots, x_n)$ be the coordinates of a generic vector in V with respect to the basis B_V. Let $B_W = \{w_1, \ldots, w_m\}$ be a basis of W and let $(y_1, \ldots, y_m)$ be the coordinates of a generic vector in W with respect to the basis B_W.

Let us suppose that $f : V \to W$ is a linear transformation such that

$$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i = a_{1j} w_1 + a_{2j} w_2 + \ldots + a_{mj} w_m,$$

this is, the coordinates of $f(v_j)$ in the basis B_W are $(a_{1j}, \ldots, a_{mj})$.

The matrix expression of f in B_V of V and B_W of W is

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

We say that $A = (a_{ij})$ is the matrix of the linear transformation f with respect to the basis B_V of V and B_W of W, which is denoted by $M_f(B_V, B_W)$.

If $f : V \to V$ is an endomorphism we denote by $M_f(B_V)$ the matrix $M_f(B_V, B_V)$.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation verifying

$$f(1,1,1) = (2,2), f(0,1,1) = (1,1), f(0,0,3) = (0,3).$$

We obtain next the matrix expression of f in the standard basis B_3 of $\mathbb{R}^3$ and B_2 of $\mathbb{R}^2$.

For this purpose we compute the images of the vectors in the basis B_3.

$$f(1,0,0) = f(1,1,1) - f(0,1,1) = (2,2) - (1,1) = (1,1),$$
$$f(0,1,0) = f(0,1,1) - \frac{1}{3}f(0,0,3) = (1,1) - (0,1) = (1,0),$$
$$f(0,0,1) = \frac{1}{3}f(0,0,3) = (0,1).$$

The columns of $M_f(B_3, B_2)$ are the coordinates in the basis B_2 os such images. Thus, the matrix expression is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then $f(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_3)$ for every $(x_1, x_2, x_3) \in \mathbb{R}^3$.

KERNEL AND IMAGE

Proposition Let $f : V \longrightarrow W$ be a linear transformation. The following statements hold:

1. The image $\mathrm{Im}(f) = \{f(v) \mid v \in V\}$ is a vector subspace of W, which is called the image of the linear transformation f.

2. If $\{v_1, \ldots, v_n\}$ is a generating set of V then $\{f(v_1), \ldots, f(v_n)\}$ is a generating set of $\mathrm{Im}(f)$. We call rank of f to the dimension of $\mathrm{Im}(f)$

$$\mathrm{rank}(f) = \dim(\mathrm{Im}(f)).$$

3. We define the kernel of the linear transformation f as the set of vectors in V whose image through f is the zero vector 0_W,

$$\mathrm{Ker}(f) = \{v \in V | f(v) = 0_W\}.$$

Then $\mathrm{Ker}(f)$ is a vector subspace of V and $\mathrm{Ker}(f) = f^{-1}(0_W)$.

Proposition Let $f : V \to W$ be a linear transformation. If V is a finitely generated vector space then:

$$\dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)) = \dim V.$$

Proposition The linear transformation $f : V \to W$ is injective if and only if

$$\mathrm{Ker}(f) = \{0_V\}.$$

Proposition Let $f : V \to W$ be a linear transformation.

1. If V is finitely generated, then f is injective if and only if $\dim V = \dim(f(V))$.
2. Let $B = \{v_1, \ldots, v_n\}$ be a basis of V. f is one-to-one if and only if $f(B) = \{f(v_1), \ldots, f(v_n)\}$ is a basis of $\mathrm{Im}(f)$ if and only if $f(B)$ is linearly independent.

Definition An isomorphism is a linear and bijective transformation $f : V \to W$. If f is an isomorphism then the vector spaces V and W are isomorphic.

Proposition

1. A linear transformation $f : V \to W$ is bijective if and only if
$$\mathrm{Im}(f) = W \text{ and } \mathrm{Ker}(f) = \{0_V\}.$$

2. An endomorphism $f : V \to V$ is bijective if and only if f is injective and surjective.

Example Let $h : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $h(x_1, x_2, x_3) = (2x_1 + 2x_2 - 3x_3, 3x_1 + x_2 - 2x_3)$. We obtain next the cartesian equations of $\mathrm{Ker}(h)$ and a basis of $\mathrm{Im}(h)$ in B_3 of $\mathbb{R}^3$ and B_2 of $\mathbb{R}^2$.

The matrix of h is
$$M_h(B_3, B_2) = \begin{pmatrix} 2 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix}.$$

A vector (x_1, x_2, x_3) in $\mathbb{R}^3$ belongs to $\mathrm{Ker}(h)$ if

$$\begin{pmatrix} 2 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

equivalently

Cartesian equations of $\mathrm{Ker}(h)$ $\left\{ \begin{array}{l} 2x_1 + 2x_2 - 3x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 0, \end{array} \right.$

Also

$$\mathrm{Im}(h) = \langle h(1,0,0), h(0,1,0), h(0,0,1) \rangle =$$
$$= \langle (2,3), (2,1), (-3,-2) \rangle = \langle (2,3), (2,1) \rangle.$$

Then $\{(2,3), (2,1)\}$ is a basis of $\mathrm{Im}(h)$.

Theorem Let us suppose that $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

Example Let V be a real vector space with $\dim V = 3$. Let $B = \{u_1, u_2, u_3\}$ be a basis of V and let $B_3 = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. The linear transformation $f : V \to \mathbb{R}^3$ determined by the conditions $f(u_1) = e_1$, $f(u_2) = e_2$ and $f(u_3) = e_3$ is an isomorphism. Then V and $\mathbb{R}^3$ are isomorphic.

OPERATIONS WITH LINEAR TRANSFORMATIONS

Let B_V and B_W be basis of V and W respectively. Given two linear transformations $f : V \to W$, $g : V \to W$ and a scalar $\lambda \in \mathbb{R}$, we define the following operations between linear transformations:

1. Sum $f + g : V \to W$ given by $(f+g)(v) = f(v) + g(v)$ for every $v \in V$.
2. Multiplication by a scalar $\lambda f : V \to W$ given by $(\lambda f)(v) = \lambda f(v)$ for every $v \in V$.

It holds that

$$M_{f+g}(B_V, B_W) = M_f(B_V, B_W) + M_g(B_V, B_W),$$
$$M_{\lambda f}(B_V, B_W) = \lambda M_f(B_V, B_W).$$

Let U be a real vector space and B_U a basis of U. Given two linear transformations $f : V \to W$ and $g : W \to U$ the composition $g \circ f : V \to U$ is a linear transformation with matrix

$$M_{g\circ f}(B_V, B_U) = M_g(B_W, B_U)M_f(B_V, B_W).$$

If V is finitely generated, a linear transformation $f : V \to W$ is an isomorphism if and only if

$$\dim V = \dim(\mathrm{Im}(f)) = \dim W.$$

Then, if $\dim V = n$ the matrix $M_f(B_V, B_W)$ is squared of size $n \times n$ and

$$\mathrm{rank}(M_f(B_V, B_W)) = \mathrm{rank}(\mathrm{Im}(f)) = \dim(\mathrm{Im}(f)) = n,$$

$M_f(B_V, B_W)$ has nonzero determinant and therefore it is an invertible matrix.

The inverse $f^{-1}: W \to V$ is also an isomorphism with matrix

$$M_{f^{-1}}(B_W, B_V) = (M_f(B_V, B_W))^{-1}.$$

Example Let us consider the linear transformations $f, g : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(x, y, z) = (4x - y, z + x, x), \qquad g(x, y, z) = (y, 2z + 3x, z).$$

We obtain next the matrices of $f - 2g$, $f \circ g$, $g \circ f$ with respect to the standard basis B of $\mathbb{R}^3$.

$$M_f(B) = \begin{pmatrix} 4 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_g(B) = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{f-2g}(B) = M_f(B) - 2M_g(B) = \begin{pmatrix} 4 & -3 & 0 \\ -5 & 0 & -3 \\ 1 & 0 & -2 \end{pmatrix},$$

$$M_{f\circ g}(B) = M_f(B)M_g(B) = \begin{pmatrix} -3 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$M_{g\circ f}(B) = M_g(B)M_f(B) = \begin{pmatrix} 1 & 0 & 1 \\ 14 & -3 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This shows that the composition of linear transformations is not commutative in general.
Furthermore f and g are invertible linear transformations since

$$\det(M_f(B)) \neq 0 \text{ and } \det(M_g(B)) \neq 0.$$

The inverse linear transformations $f^{-1}, g^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ have matrices

$$M_{f^{-1}}(B) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 4 \\ 0 & 1 & -1 \end{pmatrix}, \quad M_{g^{-1}}(B) = \begin{pmatrix} 0 & 0 & -2/3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

VECTOR INTERPRETATION OF A SYSTEM OF LINEAR EQUATIONS

Given a system of linear equations in the variables $x_1, x_2, \ldots, x_n$

$$\left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right.$$

where $m \geq 1$, $a_{ij}, b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and matrix equation $AX = b$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

with $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $b \in \mathcal{M}_{m\times 1}(\mathbb{R})$.

A solution $(s_1, \ldots, s_n)$ of the system verifies

$$s_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + s_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + s_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

That is $(b_1, \ldots, b_m)$ is a linear combination of the column vectors of the coefficient matrix of the system.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation with matrix A in the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$. The following items hold:

1. $(b_1, \ldots, b_m) \in \mathrm{Im}(f) \Leftrightarrow AX = b$ has a solution.
2. Let $S = (s_1, \ldots, s_n) \in \mathbb{R}^n$ be a solution of $AX = b$. The solution set of $AX = b$ equals the set of vectors whose image is $(b_1, \ldots, b_m)$

   $$f^{-1}(b_1, \ldots, b_m) = S + \mathrm{Ker}(f),$$

   where $\mathrm{Ker}(f)$ is the set of solutions of the system $AX = 0$.

Example Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation whose matrix in the standard basis of $\mathbb{R}^4$ and $\mathbb{R}^2$ is

$$A = \begin{pmatrix} 2 & 0 & -1 & 2 \\ 1 & -3 & 2 & 1 \end{pmatrix}.$$

The vectors whose image is $v = (1, -1) \in \mathbb{R}^2$ are the vectors in the set $f^{-1}(v)$, the solutions of

$$\begin{pmatrix} 2 & 0 & -1 & 2 \\ 1 & -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

that is

$$(0, -1/3, -1, 0) + \langle (3, 5, 6, 0), (0, 5, 6, 3) \rangle$$

where $(0, -1/3, -1, 0)$ is a vector of $f^{-1}(v)$ and

$$\mathrm{Ker}(f) = \langle (3, 5, 6, 0), (0, 5, 6, 3) \rangle.$$

CHANGE OF COORDINATES

Assume that $\dim V = n$. Let $B = \{v_1, \ldots, v_n\}$ and $B' = \{v'_1, \ldots, v'_n\}$ be bases of V. Given a vector $v \in V$, denote by $(x_1, \ldots, x_n)_B$ its coordinates w.r.t. B and $(x'_1, \ldots, x'_n)_{B'}$ its coordinates w.r.t. B'.

Assume that the coordinates of the vectors of the basis B' w.r.t. the basis B are known, namely

$$\begin{aligned} v'_1 &= a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n, \\ v'_2 &= a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n, \\ &\vdots \\ v'_1 &= a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{nn}v_n. \end{aligned}$$

Then, it holds

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

The matrix (a_{ij}) is the change of coordinates matrix from B' to B: the matrix whose columns are the coordinates of the vectors in B' w.r.t. B. It is denoted by $M(B', B)$.

The change of coordinates matrix from B to B' is denoted by $M(B, B')$ and it verifies

$$M(B, B') = M(B', B)^{-1}$$

then

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Remark The matrix $M(B', B)$ is the matrix of an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example Let B be the standard basis of $\mathbb{R}^3$ and let

$$B' = \{(1,3,0),(1,0,2),(0,4,-2)\}$$

be another basis of $\mathbb{R}^3$. Then, the matrix of the change of coordinates from B' to B is

$$M(B',B) = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 4 \\ 0 & 2 & -2 \end{pmatrix}.$$

Let v be a vector with coordinates $(1,1,2)_{B'}$, its coordinates in the basis B are $(2,11,-2)_B$ and they are obtained by:

$$\begin{pmatrix} 2 \\ 11 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 4 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

EQUIVALENT MATRICES

Let us assume that $\dim V = n$ and $\dim W = m$.

Proposition Let $f : V \to W$ be a linear transformation. Given B_V and B'_V of V, and B_W and B'_W of W, it holds

$$M_f(B'_V, B'_W) = M(B_W, B'_W)M_f(B_V, B_W)M(B'_V, B_V).$$

Definition Two matrices $A, A' \in \mathcal{M}_{m\times n}(\mathbb{R})$ are equivalent if there exist invertible matrices $P \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $Q \in \mathcal{M}_{m\times m}(\mathbb{R})$ such that

$$A' = Q^{-1}AP.$$

Equivalently, A and A' are matrices of the same linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ in different basis.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the matrix of a linear transformation in the standard basis B_3 of $\mathbb{R}^3$ and B_2 of $\mathbb{R}^2$

$$M_f(B_3, B_2) = \begin{pmatrix} 3 & 0 & -2 \\ -1 & 4 & 5 \end{pmatrix}.$$

Also consider $B_3' = \{(1,3,0),(1,0,2),(0,4,-2)\}$ of $\mathbb{R}^3$ and $B_2' = \{(2,1),(4,3)\}$ of $\mathbb{R}^2$.

We have

$$M_f(B_3', B_2') = M(B_2, B_2')M_f(B_3, B_2)M(B_3', B_3),$$

This is

$$M_f(B_3', B_2') = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 & -2 \\ -1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 4 \\ 0 & 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -35/2 & -39/2 & -6 \\ 19/2 & 19/2 & 4 \end{pmatrix}.$$

Proposition Let $f : V \to V$ be an endomorphism. Given bases B_V and B'_V of V. Then

$$M_f(B'_V) = M(B_V, B'_V)M_f(B_V)M(B'_V, B_V).$$

Two matrices $A, A' \in \mathcal{M}_{n\times n}(\mathbb{R})$ verifying

$$A' = P^{-1}AP,$$

for some invertible matrix $P \in \mathcal{M}_{n\times n}(\mathbb{R})$, are matrices of an endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ in two different bases.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be an endomorphism whose matrix in the standard basis B of $\mathbb{R}^3$ is

$$M_f(B) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

We also consider the basis $B' = \{(1,1,0),(1,0,1),(0,1,1)\}$ of $\mathbb{R}^3$. Then

$$M_f(B') = M(B',B)^{-1} M_f(B) M(B',B),$$

that is

$$M_f(B') = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 & 1 & -1/2 \\ 1/2 & 2 & 3/2 \\ -1/2 & 0 & 5/2 \end{pmatrix}.$$