8. DIAGONALIZACION

MOTIVATION

The squared matrix

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

and the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

verify

$$D = P^{-1}AP \text{ where } P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let V be a real vector space.

If $f : V \to V$ is an endomorphism, whose matrix D in some basis of V is diagonal, then numerous problems related with f are significantly simplified.

Some examples are: classify f; obtain its invariants; compute f^n, $n \in \mathbb{N}$.

Given a squared matrix A (associated to an endomorphism) we explain next how to obtain a diagonal matrix D related with A by

$$D = P^{-1}AP \text{ where } P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

EIGENVALUES AND EIGENVECTORS OF AN ENDOMORPHISM

Let V be a nonzero real vector space and $f : V \to V$ an endomorphism.

Definition A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of f if there exists a nonzero vector $v \in V$ such that

$$f(v) = \lambda v.$$

Definition If λ is an eigenvalue of f, a vector $v \in V$ verifying $f(v) = \lambda v$ is called eigenvector of f associated to λ.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism defined by

$$f(x, y, z) = (x + 3y + 3z, -3x - 5y - 3z, 3x + 3y + z).$$

Then, $\lambda = -2$ is an eigenvalue of f because there exists a nonzero vector $v = (-1, 1, 0)$ such that $f(v) = -2v$. Thus v is an eigenvector of f associated to $\lambda = -2$.

Proposition The set of all the eigenvectors of f associated to the same eigenvalue λ of f

$$V_\lambda = \{v \in V \mid f(v) = \lambda v\}$$

is a vector subspace of V called **eigenspace** of f associated to λ.

Remarks Let $id : V \to V$ be the identity map in V.

1. $V_\lambda = \mathrm{Ker}(f - \lambda id)$.
2. $\lambda \in \mathbb{R}$ is an eigenvalue of $f \Leftrightarrow$ the endomorphism $f - \lambda id$ is not one-to-one.
3. In particular, $\lambda = 0$ is an eigenvalue of $f \Leftrightarrow f$ is not one-to-one $\Leftrightarrow \mathrm{Ker}(f) = V_0 \neq \{0_V\}$.
4. Some endomorphisms do not have eigenvalues, so they do not have eigenvectors either.

Proposition Let $\lambda_1, \ldots, \lambda_p$ be p different eigenvalues of f.

1. Let v_i be a nonzero eigenvector of f associated to λ_i, $i = 1, \ldots, p$ then $v_1, \ldots, v_p$ are linearly independent.
2. $V_{\lambda_1} + \cdots + V_{\lambda_p}$ is a direct sum.

Remark If $\dim V = n$ then the endomorphism f will have at most n different eigenvalues, otherwise it would have more than n linearly independent eigenvectors.

EIGENVALUES AND EIGENVECTORS OF A SQUARED MATRIX

Let A be a squared $n \times n$ matrix whose entries are real numbers, that is $A \in \mathcal{M}_{n\times n}(\mathbb{R})$.

Definition A scalar $\lambda \in \mathbb{R}$ is an eigenvalue of A if there exists a nonzero column vector $X \in \mathcal{M}_{n\times 1}(\mathbb{R})$ such that

$$AX = \lambda X.$$

Definition If λ is an eigenvalue of A, a column vector $X \in \mathcal{M}_{n\times 1}(\mathbb{R})$ verifying $AX = \lambda X$ is called an eigenvector of A associated to λ.

Let I_n be the identity matrix of size $n \times n$.

Remarks Let us assume that $\dim V = n$ and let B be a basis of V. Let $f : V \to V$ be the endomorphism whose matrix in the basis B is A. The following statements are verified:

1. λ is an eigenvalue of $f \Leftrightarrow \lambda$ is an eigenvalue of $A \Leftrightarrow \det(A-\lambda I_n) = 0$.
2. Let $v \in V$ and let X be the column vector of the coordinates of v in the basis B, that is $X \in \mathcal{M}_{n\times 1}(K)$. Then, v is an eigenvector of $f \Leftrightarrow X$ is an eigenvector of $A \Leftrightarrow (A-\lambda I_n)X = 0$.
3. $\dim V_\lambda = n - \mathrm{rank}(A - \lambda I_n)$.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the endomorphism defined by

$$f(x, y, z) = (3x, x + 2y, 4x + 2z).$$

Let B be the standard basis of $\mathbb{R}^3$. The matrix of f in the basis B is

$$A = M_f(B) = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{pmatrix}.$$

Since $\operatorname{rank}(A - 3I_3) = 2$ the system $(A - 3I_3)X = 0$, that is

$$\begin{pmatrix} 3-3 & 0 & 0 \\ 1 & 2-3 & 0 \\ 4 & 0 & 2-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nonzero solution. Therefore, $\lambda = 3$ is an eigenvalue of f and A. The eigenspace V_3 has $\dim V_3 = 1$ and cartesian equations

$$\text{Cartesian equations of } V_3 \begin{cases} x_1 + x_2 = 0 \\ 4x_1 - x_3 = 0 \end{cases}.$$

Thus $V_3 = \{(a, a, 4a) \mid a \in \mathbb{R}\}$.
On the other hand, $\operatorname{rank}(A - 2I_3) = 1$, so $\lambda = 2$ is an eigenvalue of f and A, $\dim V_2 = 2$. Solving $(A - 2I_3)X = 0$, that is

$$\begin{pmatrix} 3-2 & 0 & 0 \\ 1 & 2-2 & 0 \\ 4 & 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we have $V_2 = \{(0, a, b) \mid a, b \in \mathbb{R}\}$ and the cartesian equation of V_2 is $x_1 = 0$.

ROOTS OF POLYNOMIALS AND THEIR MULTIPLICITY

Let $\mathbb{R}[x]$ be the set of all polynomials in x with real coefficients.

Definition Let $p(x)$ be a polynomial in $\mathbb{R}[x]$. A scalar $\lambda \in \mathbb{R}$ is a root of $p(x)$ if $p(\lambda) = 0$.
Equivalently, λ is a root of $p(x)$ if and only if $(x-\lambda)$ divides $p(x)$, that is, there exists a polynomial $q(x) \in \mathbb{R}[x]$ such that

$$p(x) = (x - \lambda)q(x).$$

Definition Let λ be a root of the polynomial $p(x)$. We call multiplicity of λ to the highest natural number m such that $(x - \lambda)^m$ divides $p(x)$, so

$$p(x) = (x - \lambda)^m q(x), \ \ q(x) \in \mathbb{R}[x].$$

Every polynomial of degree greater or equal than one has all its roots in $\mathbb{C}$. On the other hand, there exist polynomials in $\mathbb{R}[x]$ with no roots in $\mathbb{R}$. As an example, $x^2 + 1$ has real coefficients but only complex roots.

Proposition Let $p(x) \in \mathbb{R}[x]$ be a polynomial of degree n whose roots are real numbers $\lambda_1, \ldots, \lambda_p$ with multiplicities $m_1, \ldots, m_p$ respectively. Then:

1. There exists $q(x) \in \mathbb{R}[x]$ such that $p(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_p)^{m_p} q(x)$ so $m_1 + \cdots + m_p \leq n$.
2. Given $p(x) \in \mathbb{C}[x]$ all its roots are in $\mathbb{C}$ so

$$p(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_p)^{m_p} \text{ with } m_1 + \cdots + m_p = n.$$

CHARACTERISTIC POLYNOMIAL

Definition Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. The **characteristic polynomial** of A is $\det(A - \lambda I_n)$ and its **characteristic equation** is

$$\det(A - \lambda I_n) = 0.$$

Proposition Let A and A' be matrices in $\mathcal{M}_{n\times n}(\mathbb{R})$. If A and A' are matrices of the same endomorphism in different basis then they have the same characteristic polynomial.

Let V be a nonzero real vector space and let $f : V \to V$ be an endomorphism.

Remark

1. All the matrices associated to f in different bases of V have the same characteristic polynomial.
2. The converse of the proposition is not true.

Definition The characteristic polinomial of f is the characteristic polynomial of any of the matrices associated to f in the different bases of V. Analogously with the characteristic equation of f.

Example Let us compute the characteristic polynomial of the endomorphism f of $\mathbb{R}^5$ defined by

$$f(x_1, x_2, x_3, x_4, x_5) = (-x_2, x_1 + x_3, 2x_3 - x_4, 2x_4 + 6x_5, 3x_5).$$

Let B be the standard basis of $\mathbb{R}^5$ and $A = M_f(B)$, then:

$$\det(A-\lambda I_5) = \begin{vmatrix} 0-\lambda & -1 & 0 & 0 & 0 \\ 1 & 0-\lambda & 1 & 0 & 0 \\ 0 & 0 & 2-\lambda & -1 & 0 \\ 0 & 0 & 0 & 2-\lambda & 6 \\ 0 & 0 & 0 & 0 & 3-\lambda \end{vmatrix} =$$

$$=(3-\lambda)(2-\lambda)^2 \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix}$$

$$=(3-\lambda)(2-\lambda)^2(\lambda^2+1).$$

MULTIPLICITY OF AN EIGENVALUE

Let V be a nonzero real vector space and $f : V \to V$ an endomorphism. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$.

Definition Let λ be an eigenvalue of f (or A). We call multiplicity of λ to its multiplicity as a root of the characteristic equation of f (or A).

Theorem Let λ be an eigenvalue of f (or A) with multiplicity m. Then

$$1 \leq \dim V_\lambda \leq m.$$

Remarks

1. Let λ be an eigenvalue of f (or A) with multiplicity m. If $m = 1$ then $\dim V_\lambda = 1$

2. If $A, A' \in \mathcal{M}_{n\times n}(\mathbb{R})$ are matrices of the same endomorphism in different bases then they have the same eigenvalues, with the same multiplicities and the same dimensions of their eigenspaces.

Proposition Let us suppose that $\dim V = n$. Let $\lambda_1, \ldots \lambda_p$ be the distinct eigenvalues of f (or A), $m_1, \ldots, m_p$ their multiplicities and $d_1, \ldots, d_p$ the dimensions of the corresponding vector subspaces. Then the maximum number of linearly independent eigenvectors of f (or A) is $d_1 + \cdots + d_p$. Furthermore,

$$p \leq d_1 + \cdots + d_p \leq m_1 + \cdots + m_p \leq n.$$

Remark The characteristic polynomial may not have only real roots and then $m_1 + \cdots + m_p < n$. In fact, it could have no real roots and therefore no eigenvalues in this case.

Example We obtain next the eigenvalues of the matrix A together with their multiplicities and the dimensions of the eigenspac

$$A = \begin{pmatrix} 1 & 2 & 10 \\ 2 & 1 & 10 \\ -1 & -1 & -6 \end{pmatrix}.$$

The characteristic polynomial is $\det(A - \lambda I_3) = \lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 2)(\lambda + 1)^2$. Then we have eigenvalues $\lambda_1 = -2$ with multiplicity $m_1 = 1$ and $\lambda_2 = -1$ with multiplicity $m_2 = 2$. The dimensions are

$$d_1 = \dim V_{\lambda_1} = 3 - \operatorname{rank}(A - \lambda_1 I_3) = 1,$$
$$d_2 = \dim V_{\lambda_2} = 3 - \operatorname{rank}(A - \lambda_2 I_3) = 2.$$

DIAGONALIZATION OF ENDOMORPHISMS AND SQUARED MATRICES

Let V be a nonzero real vector space and let $f : V \to V$ be an endomorphism. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$.

Definition The endomorphism f is **diagonalizable** if there exists a basis B' of V so that the matrix of f, $M_f(B')$ is diagonal. Then, to **diagonalize** f is to find B'.

Definition The matrix A is **diagonalizable** if there exists a diagonal matrix D and an invertible matrix $P \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $D = P^{-1}AP$. Then, to **diagonalize** A means to find D and P.

Remarks

1. Let us suppose that A is the matrix of f in a basis B. Then:

$$f \text{ is diagonalizable } \Leftrightarrow A \text{ is diagonalizable.}$$

2. If $D = P^{-1}AP$ is a diagonalization of A then

   a) $D = M_f(B')$ is the matrix of f in a basis B' of V consisting of eigenvectors.

   b) $P = M(B', B)$ is the matrix of the change of coordinates from B' to B.

Proposition An endomorphism f is diagonalizable if and only if there exists a basis of V consisting of eigenvectors of f.

Theorem Let us suppose that $\dim V = n$. Let $\lambda_1, \ldots, \lambda_p$ be the distinct eigenvalues of f (or A), $m_1, \ldots, m_p$ their multiplicities and $d_1, \ldots, d_p$ the dimensions of the corresponding subspaces. The necessary and sufficient conditions for the existence of a basis of V consisting of eigenvectors are:

1. The characteristic polynomial of f has only real roots

   $$\lambda_1, \ldots, \lambda_p$$

   , that is

   $$m_1 + \ldots + m_p = n.$$

2. The multiplicity of each eigenvalue equals the dimension of its eigenspace, this is

   $$m_i = d_i, \quad i = 1, \ldots, p.$$

Corollary Let us suppose that f is diagonalizable. Let $\lambda_1, \dots, \lambda_p$ be the eigenvalues of f with multiplicities $m_1, \dots, m_p$ respectively. Let B_i be a basis of the eigenspace V_{λ_i} having $m_i = d_i$ elements, $i = 1, \dots, p$. Then:

1. $B' = B_1 \cup \cdots \cup B_p$ is a basis of V consisting of eigenvectors of f.
2. The matrix of f in the basis B' is diagonal and its main diagonal contains the elements

$$\lambda_1, \overset{m_1}{\ldots}, \lambda_1, \lambda_2, \overset{m_2}{\ldots}, \lambda_2, \dots, \lambda_p, \overset{m_p}{\ldots}, \lambda_p$$

Examples Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be endomorphism defined by

$$f(x, y, z) = (x + 2y + 10z, 2x + y + 10z, -x - y - 6z)$$

whose matrix in the standard basis B of $\mathbb{R}^3$ is a matrix A whose eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -1$. A basis of V_{λ_1} is

$$B_{\lambda_1} = \{(-2, -2, 1)\}$$

and of V_{λ_2} is

$$B_{\lambda_2} = \{(-5, 0, 1), (-1, 1, 0)\}.$$

Then a basis of $\mathbb{R}^3$ consisting of eigenvectors of f is

$$B' = B_{\lambda_1} \cup B_{\lambda_2} = \{(-2, -2, 1), (-5, 0, 1), (-1, 1, 0)\}.$$

Finally,

$$D = P^{-1}AP = M_f(B') = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where

$$P = \begin{pmatrix} -2 & -5 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$