CHAPTER II: AFFINE AND EUCLIDEAN GEOMETRY

2. AFFINE TRANSFORMATIONS

2.1 Definition and first properties

Definition

Let $(\mathbb{A}, V, \phi)$ and $(\mathbb{A}', V', \phi')$ be two real affine spaces. We will say that a map

$$f \colon \mathbb{A} \longrightarrow \mathbb{A}'$$

is an *affine transformation* if there exists a linear transformation $\bar{f} \colon V \longrightarrow V'$ such that:

$$\bar{f}(\overline{PQ}) = \overline{f\,(P)\,f\,(Q)}, \quad \forall P, Q \in \mathbb{A}.$$

This is equivalent to say that for every $P \in \mathbb{A}$ and every vector $\bar{u} \in V$ we have

$$f(P + \overline{u}) = f(P) + \bar{f}(\overline{u}).$$

We call $\bar{f}$ *the linear transformation associated to* f.

Proposition

Let $(\mathbb{A}, V, \phi)$ and $(\mathbb{A}', V', \phi')$ be two affine subspaces and let $f\colon \mathbb{A} \longrightarrow \mathbb{A}'$ be an affine transformation with associated linear transformation $\bar{f}\colon V \longrightarrow V'$. The following statements hold:

1. f is injective if and only if $\bar{f}$ is injective.
2. f is surjective if and only if $\bar{f}$ is surjective.
3. f is bijective if and only if $\bar{f}$ is bijective.

Proposition

Let $g\colon \mathbb{A} \longrightarrow \mathbb{A}'$ and $f\colon \mathbb{A}' \longrightarrow \mathbb{A}''$ be two affine transformations. The composition $f \circ g\colon \mathbb{A} \longrightarrow \mathbb{A}''$ is also an affine transformation and its associated linear transformation is $\overline{f \circ g} = \bar{f} \circ \bar{g}$.

Proposition

Let $f, g\colon \mathbb{A} \longrightarrow \mathbb{A}'$ be two affine transformations which coincide over a point P, $f(P) = g(P)$, and which have the same associated linear transformation $\bar{f} = \bar{g}$. Then $f = g$.

Proof

Every $X \in \mathbb{A}$ verifies:

$$f(X) = f(P + \overline{PX}) = f(P) + \bar{f}(\overline{PX}) = g(P) + \bar{g}(\overline{PX}) = g(X).$$

2.2 Matrix associated to an affine transformation

Let $(\mathbb{A}, V, \phi)$ and $(\mathbb{A}', V', \phi')$ be two affine subspaces and let $f\colon \mathbb{A} \longrightarrow \mathbb{A}'$ be an affine transformation with associated linear transformation $\bar{f}\colon V \longrightarrow V'$. We consider affine coordinate systems $\mathcal{R} = \{O; B\}$, $B = \{\overline{e}_1, \ldots, \overline{e}_n\}$ and $\mathcal{R}' = \{O'; B'\}$, $B' = \{\overline{e}'_1, \ldots, \overline{e}'_m\}$ of the spaces $\mathbb{A}$, $\mathbb{A}'$ respectively. Let us assume that:

$$\overline{O'f(O)} = b_1\overline{e}'_1 + \cdots + b_m\overline{e}'_m,$$

$$\left\{ \begin{array}{c} \bar{f}(\overline{e}_1) = a_{11}\overline{e}'_1 + \cdots + a_{m1}\overline{e}'_m \\ \vdots \\ \bar{f}(\overline{e}_n) = a_{1n}\overline{e}'_1 + \cdots + a_{mn}\overline{e}'_m \end{array} \right.$$

Let P be the point with coordinates $(x_1, \ldots, x_n)$ with respect to $\mathcal{R}$ and let $(y_1, \ldots, y_m)$ be the coordinates of $f(P) \in \mathbb{A}'$.

Then:

$$\begin{pmatrix} 1 \\ y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ b_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

We will write

$$M_f(\mathcal{R}, \mathcal{R}') = \begin{pmatrix} 1 & \overline{0}^t \\ \overline{b} & M_{\bar{f}}(B, B') \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ b_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m & a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

where $\overline{b}$ are the coordinates of $f(O)$ in the coordinate system $\mathcal{R}'$ and $M_{\bar{f}}(B, B')$ is the matrix associated to the linear transformation $\bar{f}$ taking in V the basis B and in V' the basis B'.

Example 1

Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B = \{\overline{e}_1, \overline{e}_2, \overline{e}_3\}\}$, and let $(\mathbb{A}', V', \phi')$ be an affine space with affine coordinate system $\mathcal{R}' = \{O'; B' = \{\overline{e}'_1, \overline{e}'_2\}\}$. Is the transformation $f\colon \mathbb{A} \longrightarrow \mathbb{A}'$, $f(x, y, z) = (x - 2y + 5, x - z + 1)$ an affine transformation? Give its associated linear transformation and obtain the matrix associated to f in the coordinate systems $\mathcal{R}, \mathcal{R}'$.

<u>Solution.</u>

To see if f is an affine transformation we have to check if there exists a linear transformation $\bar{f}\colon V \longrightarrow V'$ such that $\overline{f(P)f(Q)} = \bar{f}\left(\overline{PQ}\right)$ for every pair of points $P, Q \in \mathbb{A}$. We take $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then

$$\overline{PQ} = Q - P = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

and

$$\begin{aligned}
\overline{f(P)f(Q)} &= f(Q) - f(P) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1) \\
&= (x_2 - 2y_2 + 5, x_2 - z_2 + 1) - (x_1 - 2y_1 + 5, x_1 - z_1 + 1) \\
&= ((x_2 - x_1) - 2\,(y_2 - y_1)\,, (x_2 - x_1) - (z_2 - z_1)) \\
&= \bar{f}\,(x_2 - x_1, y_2 - y_1, z_2 - z_1)\,.
\end{aligned}$$

Therefore, f is an affine transformation and its associated linear transformation is $\bar{f}\,(x, y, z) = (x - 2y, x - z)$.

The coordinates of the origin O in the coordinate system $\mathcal{R}$ are the cooordinates of the vector $\overline{OO} = (0, 0, 0)$ in the basis B where $\overline{e}_1 = (1, 0, 0)_B$, $\overline{e}_2 = (0, 1, 0)_B$ and $\overline{e}_3 = (0, 0, 1)_B$.

Then:

$$\begin{aligned} f(O) &= f(0,0,0) = (5,1), \\ \bar{f}(\overline{e}_1) &= \bar{f}(1,0,0) = (1,1), \\ \bar{f}(\overline{e}_2) &= \bar{f}(0,1,0) = (-2,0), \\ \bar{f}(\overline{e}_3) &= \bar{f}(0,0,1) = (0,-1). \end{aligned}$$

So,

$$M_f(\mathcal{R},\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & -2 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}.$$

Example 2

Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B = \{\overline{e}_1, \overline{e}_2\}\}$, and let $(\mathbb{A}', V', \phi')$ be an affine space with affine coordinate system $\mathcal{R}' = \{O'; B' = \{\overline{e}'_1, \overline{e}'_2, \overline{e}'_3\}\}$. Determine the affine transformation $f\colon \mathbb{A} \longrightarrow \mathbb{A}'$, such that

$$\begin{aligned} f(1,2) &= (1,2,3), \\ \bar{f}(\overline{e}_1) &= \overline{e}'_1 + 4\overline{e}'_2, \\ \bar{f}(\overline{e}_2) &= \overline{e}'_1 - \overline{e}'_2 + \overline{e}'_3. \end{aligned}$$

Find the matrix associated to f in the coordinate systems $\mathcal{R}, \mathcal{R}'$.

Solution.

We know the value of f at the point $P(1,2)$ and the linear transformation associated to f, therefore we can determine f.

$$\begin{aligned} \bar{f}(1,0) &= (1,4,0)_{B'}, \\ \bar{f}(0,1) &= (1,-1,1)_{B'}, \end{aligned}$$

To calculate the matrix associated to f we need to compute $f(O)$. We have:

$$\begin{aligned} f(P) \underset{f \text{ is affine}}{=} f(O)+\bar{f}(\overline{OP}) &= f(O)+\bar{f}(1\overline{e}_1+2\overline{e}_2) \\ &\underset{\bar{f} \text{ is linear}}{=} f(O)+\bar{f}(\overline{e}_1)+2\bar{f}(\overline{e}_2) \\ &= f(O)+(1,4,0)+2(1,-1,1) \end{aligned}$$

so

$$\begin{aligned} f(O) &= f(1,2)-(1,4,0)-2(1,-1,1) \\ &= (1,2,3)-(1,4,0)-2(1,-1,1) \\ &= (-2,0,1)\,. \end{aligned}$$

therefore

$$M_f(\mathcal{R},\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

As

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ x_1 + x_2 - 2 \\ 4x_1 - x_2 \\ x_2 + 1 \end{pmatrix}$$

we have:

$$f(x_1, x_2) = (x_1 + x_2 - 2, 4x_1 - x_2, x_2 + 1)\,.$$

Example 3

Let $(\mathbb{R}^2, \mathbb{R}^2, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B = \{\overline{e}_1, \overline{e}_2\}\}$. Determine the affine transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$f(1,1) = (7,5),\ \ f(1,2) = (11,4),\ \ f(2,1) = (8,8).$$

Solution.

To give an affine transformation $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ we need three points that form an affine coordinate system and their images.

First method

Since $\overline{P_0P_1} = (0,1)$ and $\overline{P_0P_2} = (1,0)$, then we know that

$$\begin{aligned}\bar{f}(\overline{e}_1) &= \bar{f}(1,0) = \bar{f}(\overline{P_0P_2}) = f(P_2) - f(P_0) = (1,3),\\ \bar{f}(\overline{e}_2) &= \bar{f}(0,1) = \bar{f}(\overline{P_0P_1}) = f(P_1) - f(P_0) = (4,-1).\end{aligned}$$

Also $\overline{OP_0} = \overline{e}_1 + \overline{e}_2$ so we have:

$$\begin{aligned}f(P_0) &= f(O) + \bar{f}(\overline{OP_0}) = f(O) + \bar{f}(\overline{e}_1 + \overline{e}_2)\\ &= f(O) + \bar{f}(\overline{e}_1) + \bar{f}(\overline{e}_2)\end{aligned}$$

then

$$\begin{aligned} f(O) &= f(P_0) - \bar{f}(\overline{e}_1) - \bar{f}(\overline{e}_2) = (7,5) - (1,3) - (4,-1) \\ &= (2,3). \end{aligned}$$

finally,

$$M_f(\mathcal{R},\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \\ 3 & 3 & -1 \end{pmatrix}$$

and $f(x_1,x_2) = (2 + x_1 + 4x_2,\ 3 + 3x_1 - x_2)$.

Second method

The set $\mathcal{R}' = \{P_0(1,1), P_1(1,2), P_2(2,1)\}$ is an affine coordinate system since $\overline{P_0P_1} = (0,1)$ and $\overline{P_0P_2} = (1,0)$ is a basis of $\mathbb{R}^2$. We have:

$$\begin{aligned} f(P_0) &= f(1,1) = (7,5), \\ \bar{f}(\overline{P_0P_1}) &= \overline{f(P_0)f(P_1)} = f(P_1) - f(P_0) = (11,4) - (7,5) = (4,-1), \\ \bar{f}(\overline{P_0P_2}) &= \overline{f(P_0)f(P_2)} = f(P_2) - f(P_0) = (8,8) - (7,5) = (1,3). \end{aligned}$$

Therefore,

$$M_f(\mathcal{R},\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 7 & 4 & 1 \\ 5 & -1 & 3 \end{pmatrix}$$

To obtain $M_f(\mathcal{R},\mathcal{R})$, we are going to change the coordinate system from $\mathcal{R}'$ to $\mathcal{R}$:

$$\begin{aligned} M_f(\mathcal{R},\mathcal{R}) &= M_f(\mathcal{R}',\mathcal{R})M(\mathcal{R},\mathcal{R}') = M_f(\mathcal{R}',\mathcal{R})M(\mathcal{R}',\mathcal{R})^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 7 & 4 & 1 \\ 5 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \\ 3 & 3 & -1 \end{pmatrix} \end{aligned}$$

Therefore $f(x_1, x_2) = (2 + x_1 + 4x_2, 3 + 3x_1 - x_2)$.

Example 4

Determine the affine transformation $f\colon \mathbb{A}_3 \longrightarrow \mathbb{A}_3$ which transforms the points $P_0(0,0,0)$, $P_1(0,1,0)$, $P_2(1,1,1)$ and $P_3(1,1,4)$ in the points $Q_0(2,0,2)$, $Q_1(2,-1,1)$, $Q_2(2,1,3)$ and $Q_3(5,7,6)$ respectively.

Solution.

To give an affine transformation $f\colon \mathbb{A}_3 \longrightarrow \mathbb{A}_3$ we will need four points that form an affine coordinate system and their images.
The set $\mathcal{R}' = \{P_0(0,0,0), P_1(0,1,0), P_2(1,1,1), P_3(1,1,4)\}$ is an affine coordinate system as $\overline{P_0P_1} = (0,1,0)$, $\overline{P_0P_2} = (1,1,1)$ and $\overline{P_0P_3} = (1,1,4)$ is a basis of $\mathbb{R}^3$ because

$$\dim(\overline{P_0P_1}, \overline{P_0P_2}, \overline{P_0P_3}) = 3.$$

We have:

$$\begin{aligned}
f(P_0) &= Q_0 = (2,0,0),\\
\bar{f}(\overline{P_0P_1}) &= \overline{f(P_0)f(P_1)} = f(P_1) - f(P_0) = Q_1 - Q_0 = (0,-1,-1),\\
\bar{f}(\overline{P_0P_2}) &= \overline{f(P_0)f(P_2)} = f(P_2) - f(P_0) = Q_2 - Q_0 = (0,1,1),\\
\bar{f}(\overline{P_0P_3}) &= \overline{f(P_0)f(P_3)} = f(P_3) - f(P_0) = Q_3 - Q_0 = (3,7,4).
\end{aligned}$$

Therefore

$$M_f(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & -1 & 1 & 7 \\ 0 & -1 & 1 & 4 \end{pmatrix}$$

To obtain $M_f(\mathcal{R},\mathcal{R})$, we are going to change the coordinate system from $\mathcal{R}'$ to $\mathcal{R}$:

$$\begin{aligned} M_f(\mathcal{R},\mathcal{R}) &= M_f(\mathcal{R}',\mathcal{R})M(\mathcal{R},\mathcal{R}') = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & -1 & 1 & 7 \\ 0 & -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \end{aligned}$$

Thus, $f(x_1,x_2,x_3) = (2 - x_1 + x_3, -x_2 + 2x_3, x_1 - x_2 + x_3)$.

2.3 Affine invariant subspaces

Proposition

Let $(\mathbb{A}, V, \phi)$ and $(\mathbb{A}', V', \phi')$ two affine spaces and let $f\colon \mathbb{A} \longrightarrow \mathbb{A}'$ be an affine transformation with associated linear transformation $\bar{f}\colon V \longrightarrow V'$. The following statements hold:

1. If $L \subset \mathbb{A}$ is an affine subspace of $\mathbb{A}$ then
$$f(L) = \{P' \in \mathbb{A}' \mid \text{there exists } P \in L \text{ such that } f(P) = P'\}$$
is an affine subspace of $\mathbb{A}'$.

2. If $L' \subset \mathbb{A}'$ is an affine subspace of $\mathbb{A}'$ then the set
$$L = \{P \in \mathbb{A} \mid f(P) \in L'\}$$
is an affine subspace of $\mathbb{A}$, called *the inverse image* of L' and denoted $f^{-1}(L')$.

Definition

Let $(\mathbb{A}, V, \phi)$ be an affine space and $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ an affine transformation. We will say that a point $P \in \mathbb{A}$ is a *fixed point* of f if $f(P) = P$.

Proposition

Let $(\mathbb{A}, V, \phi)$ be an affine space and $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ an affine transformation. The set of fixed points of f; this is,

$$F = \{X \in \mathbb{A} \mid f(X) = X\}$$

is an affine subspace of $\mathbb{A}$ with associated vector space the subspace of V of eigenvectors of $\bar{f}$ associated to the eigenvalue $\lambda = 1$.

Strategy to search for fixed points

Let $(\mathbb{A}, V, \phi)$ be an affine space, $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ an affine transformation, and $\mathcal{R} = \{O; B\}$ a coordinate system of $\mathbb{A}$. Let

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & \bar{0}^t \\ \bar{b} & A \end{pmatrix}$$

be the matrix associated to f, where A is the matrix associated to the linear transformation $\bar{f}$ in the basis B.

If P is a fixed point, the following statement holds:

$$\begin{aligned} P &= f(P) = f(O + \overline{OP}) = f(O) + \bar{f}(\overline{OP}) \\ &= \bar{b} + A \cdot \overline{OP} \end{aligned}$$

or equivalently,

$$\bar{0} = (A - I)\,\overline{OP} + \bar{b}$$

which is the equation that the fixed points of f must satisfy.

Example

Obtain the fixed points of the affine transformation

$$f(x, y) = (-2y + 1, x + 3y - 1).$$

Solution.

The matrix associated to f is

$$M_f(\mathcal{R}, \mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ -1 & 1 & 3 \end{pmatrix}$$

and the matrix associated to the linear transformation $\bar{f}$ is

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}.$$

The fixed points of f are the solutions $P(x, y)$ of the following matrix equation:

$$\overline{0} = (A - I)\, P + \overline{b}$$

this is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longleftrightarrow x + 2y - 1 = 0$$

Therefore, the points of the line $x + 2y - 1 = 0$ are the fixed points of f.

Definition

Let $(\mathbb{A}, V, \phi)$ be an affine space, $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ an affine transformation and S an affine subspace of $\mathbb{A}$. We will say that S is an *invariant affine subspace of* f if $f(S) \subset S$.

Observation

Let $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ be an affine transformation with associated linear transformation $\bar{f}\colon V \longrightarrow V$ and S an affine subspace of $\mathbb{A}$, which contains the point P and whose associated vector space is generated by the vectors $\overline{u}_1, \ldots, \overline{u}_r$; this is,

$$S \equiv P + \langle \overline{u}, \ldots, \overline{u}_r \rangle .$$

Then the affine subspace $f(S)$ contains the point $f(P)$ and is generated by the vectors $\bar{f}(\overline{u}_1), \ldots, \bar{f}(\overline{u}_r)$; this is,

$$f(S) = f(P) + \left\langle \bar{f}(\overline{u}_1), \ldots, \bar{f}(\overline{u}_r) \right\rangle .$$

Then S is invariant under f if and only if

1. $\left\langle f(\overline{u}_1), \ldots, \bar{f}(\overline{u}_r) \right\rangle \subset \langle u, \ldots, \overline{u}_r \rangle$
2. $\overline{Pf(P)} \in \langle \overline{u}, \ldots, \overline{u}_r \rangle$

In particular, a line $r \equiv P + \langle u \rangle$ is invariant under f if and only if

1. $\left\langle \bar{f}(\overline{u}) \right\rangle \subset \langle \overline{u} \rangle \iff \bar{f}(\overline{u}) = \lambda \overline{u}$; this is , $\bar{u}$ is an eigenvector of the linear transformation $\bar{f}$
2. $\overline{Pf(P)} \in \langle \overline{u} \rangle$

Example

Obtain the invariant subspaces of the transformation f of the former example.

Solution.

To search for the invariant subspaces of f first we compute the eigenvalues of $\bar{f}$. The characteristic polynomial A is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

and, therefore, the eigenvalues of A are $\lambda = 1, 2$.

The corresponding subspaces of eigenvectors of $\bar{f}$ are

$$\begin{aligned}
V(1) &= \{\overline{v} \mid (A - 1I)\overline{v} = \overline{0}\} \\
&= \left\{(x, y) \text{ such that } \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \\
&= \{(x, y) \text{ such that } x + 2y = 0\} = \langle (2, -1) \rangle \\
V(2) &= \{\overline{v} \mid (A - 2I)\overline{v} = \overline{0}\} \\
&= \left\{(x, y) \text{ such that } \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \\
&= \{(x, y) \text{ such that } x + y = 0\} = \langle (1, -1) \rangle
\end{aligned}$$

On the other hand

$$\begin{aligned}\overline{Pf(P)} &= f(P) - P = (-2y+1, x+3y-1) - (x,y) \\ &= (-x-2y+1, x+2y-1) \in V(2)\end{aligned}$$

as the components of vector $\overline{Pf(P)}$ satisfy the equation of $V(2)$.

Therefore, the lines with associated vector space $V(2) = \langle(1,-1)\rangle$ are invariant lines of f because

$$\begin{aligned}\bar{f}(1,-1) &= 2(1,-1) \\ \overline{Pf(P)} &\in V(2)\end{aligned}$$

If $x+2y-1 = 0$ (this is the line of the fixed points of f) then $\overline{Pf(P)} = \overline{0} \in V(1)$.

The line of fixed points, is in particular, an invariant line of f.

Example

Let $(\mathbb{A}_3, V, \phi)$ be an affine space and $\mathcal{R} = \{O; \{\overline{e}_1, \overline{e}_2, \overline{e}_3\}\}$ a coordinate system of $\mathbb{A}_3$. Determine the affine transformation $f\colon \mathbb{A}_3 \longrightarrow \mathbb{A}_3$ such that the plane $\pi \equiv x + 2y - z = 1$ is a plane of fixed points of f and the vector $\overline{e}_1$ is an eigenvector of $\bar{f}$ associated to the eigenvalue 3.

Solution.

The plane π is a plane of fixed points, any point of the plane is a fixed point of f. For example, the point $P(1,0,0) \in \pi$ is a fixed point of f; this is, $f(P) = P$. Also, we know that the vectors of the vector subspace associated to π, this is, vectors from the plane $\overline{\pi} \equiv x + 2y - z = 0$, are eigenvectors associated to the eigenvalue 1.
For example, for

$$\overline{u} = (1,0,1) \in \overline{\pi}, \overline{v} = (0,1,2) \in \overline{\pi}$$

we have: $\bar{f}(\overline{u}) = \bar{u}$ and $\bar{f}(\overline{v}) = \overline{v}$, this is,

$$\bar{f}(1,0,1) = (1,0,1) \text{ and } \bar{f}(0,1,2) = (0,1,2),$$

and, we also know that $\bar{f}(\overline{e}_1) = 3\overline{e}_1$; this is, $\bar{f}(1,0,0) = 3(1,0,0)$.
As $B' = \{\overline{e}_1, \overline{u}, \overline{v}\}$ is a basis of V, we consider the coordinate system $\mathcal{R} = \{P; B'\}$. We have:

$$M_f(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Also, we have:

$$\begin{aligned} M_f(\mathcal{R}) &= M_f(\mathcal{R}', \mathcal{R}) M_(\mathcal{R}, \mathcal{R}') = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 4 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Checking the answer. Obviously $\bar{f}(\overline{e}_1) = 3\overline{e}_1$ and it holds:

$$\begin{aligned}
f(P) &= f(1,0,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 4 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = P \\
\bar{f}(\vec{u}) &= \bar{f}(1,0,1) = \begin{pmatrix} 3 & 4 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \overline{u} \\
\bar{f}(\vec{v}) &= \bar{f}(1,0,1) = \begin{pmatrix} 3 & 4 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \overline{v}.
\end{aligned}$$

2.4 Some examples of affine transformations of a space in itself

Let $(\mathbb{A}, V, \phi)$ be an affine space and let $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ be an affine transformation with associated linear map $\bar{f}\colon V \longrightarrow V$. Let $M_f(\mathcal{R})$ be the matrix associated to f in the coordinate system $\mathcal{R}$.

2.4.1 Translations

Given a vector $\overline{v} \in V$, we define *the translation of vector* $\overline{v}$ as the transformation $T_{\overline{v}}\colon \mathbb{A} \longrightarrow \mathbb{A}$ such that $f(P) = P + \overline{v}$.

Proposition
Every translation $T_{\overline{v}}$ is an affine transformation where the associated linear map is the identity.

Proof Given $P, Q \in \mathbb{A}$ the following statement holds:

$$\begin{aligned} \overline{T_{\overline{v}}(P)T_{\overline{v}}(Q)} &= \overline{T_{\overline{v}}(P)P} + \overline{PQ} + \overline{QT_{\overline{v}}(Q)} \\ &= -\overline{v} + \overline{PQ} + \overline{v} = \overline{PQ}. \end{aligned}$$

2.4.2 Projections

An affine transformation $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ is a *projection* if $f^2 = f$. Therefore, if $M_f(\mathcal{R})$ is idempotent ($M_f(\mathcal{R})^2 = M_f(\mathcal{R})$).
The linear transformation associated to a projection is also idempotent $\bar{f}^2 = \bar{f}$.

Remark

The set of fixed points of a projection f is the affine subspace $\mathrm{Im}(f)$.

2.4.3 Homotethy

An affine transformation $f\colon \mathbb{A} \longrightarrow \mathbb{A}$ is an *homothety of ratio* r if $\bar{f} = rId_V$, where Id_V is the identity map on V.

Remark

A homotethy of ratio r has only one fixed point C called *center of the homothety*. The image of any other point P is obtained as follows:

$$f(P) = C + r\overline{CP}.$$

How to calculate the center of a homothety

Let $C \in \mathbb{A}$ be the center of the homothety f. We have:

$$C = f(C) = f(P + \overline{PC}) = f(P) + \bar{f}(\overline{PC}) = f(P) + r\overline{PC} \Longrightarrow \overline{f(P)C} = r\overline{PC}$$

We also have:

$$\overline{PC} = \overline{Pf(P)} + \overline{f(P)C} = \overline{Pf(P)} + r\overline{PC} \Longrightarrow (1-r)\overline{PC} = \overline{Pf(P)}.$$

Therefore, the fixed point C verifies

$$C = P + \frac{1}{1-r}\overline{Pf(P)}.$$

Example 1

Study whether the affine transformation $f(x,y,z) = (1+\frac{2}{3}x, -1+\frac{2}{3}y, 2+\frac{2}{3}z)$ has a fixed point or an invariant subspace.

Solution.

The matrix associated to f is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{2}{3} & 0 & 0 \\ -1 & 0 & \frac{2}{3} & 0 \\ 2 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

and the matrix associated to the linear transformation $\bar{f}$ is

$$M_{\overline{f}}(B) = \frac{2}{3} Id.$$

Therefore, f is a homothety of ratio $r = \frac{2}{3}$. The center of the homothety is:

$$C = P + \frac{1}{1-\frac{2}{3}} \overline{Pf(P)}$$

for every $P \in \mathbb{A}$. We take $P(0,0,0)$ then $f(P) = f(0,0,0) = (1,-1,2)$ and $\overline{Pf(P)} = (1,-1,2)$, therefore

$$C = \frac{3}{3-2}(1,-1,2) = (3,-3,6).$$

The invariant subspaces of f are:

- The center C since it is a fixed point

- The lines that contain the center

- The planes that contain the center

Example 2

Study whether the affine transformation $f(x, y, z) = (x + 1, y + 2, z + 3)$ has fixed points or invariant subspaces.

Solution.

The matrix associated to f is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

and the matrix associated to the linear transformation $\bar{f}$ is the identity. Therefore, f is a translation of vector $\overline{v} = \overline{Of(O)} = (1, 2, 3) - (0, 0, 0) = (1, 2, 3)$.
The translations do not have any fixed point.
The invariant subspaces of f are:

- The lines whose direction is the direction of the translation vector; this is, lines $r \equiv P + \langle \overline{v} \rangle$.

- The planes whose direction contains the translation vector; this is, planes of the form $\pi \equiv P + \langle \overline{v}, \overline{w} \rangle$.

Example 3

Study whether the affine transformation $f(x, y, z) = (-2 + 2x - y, -4 + 2x - y, z)$ has fixed points or any invariant subspaces.

Solution.

The matrix associated to f is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ -4 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the matrix associated to the linear transformation $\bar{f}$ is

$$A = M_f(B) = \begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A are the roots of:

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}=-\lambda\left(\lambda-1\right)^{2}.$$

The matrix A may be idempotent because its eigenvalues are $\lambda=0$ and 1. We can check that $A^2=A$ and therefore, A is idempotent. So f is a projection.

The fixed points of f satisfy the following equation:

$$\overline{0} = (A - I)\, P + \overline{b}$$

this is,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}$$

equivalently

$$\begin{cases} 0 = x - y - 2 \\ 0 = 2x - 2y - 4 \\ 0 = 0 \end{cases}$$

Therefore the plane π of equation $x - y - 2 = 0$ is a plane of fixed points (their associated vector space is the one of eigenvectors associated to the eigenvalue $\lambda = 1$).

The subspace of eigenvectors associated to the eigenvalue $\lambda = 0$ is:

$$
\begin{aligned}
V(0) &= \left\{ (x, y, z) \text{ such that } \begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\
&= \{(x, y, z) \text{ such that } 2x - y = 0, \; z = 0\}
\end{aligned}
$$

On the other hand

$$\begin{aligned}\overline{Pf(P)} &= f(P) - P = (-2+2x-y, -4+2x-y, z) - (x, y, z)\\ &= (-2-x-y, -4+2x-2y, 0) \in V(0)\end{aligned}$$

as the components of the vector $\overline{Pf(P)}$ verify the equation of $V(0)$. Therefore, the lines with associated vector space $V(0) = \langle (1,2,0) \rangle$, are invariant lines of f.

The invariant subspaces of f are:

- The lines with associated vector space $V(0) = \langle (1,2,0) \rangle$.

- The planes containing invariant lines.

- The plane of fixed points $\pi \equiv x - y - 2 = 0$.

- The lines contained in the plane of fixed points are lines of fixed points.

Example 4

Obtain the analytic expression of the affine tranformation $f\colon \mathbb{A}_3 \longrightarrow \mathbb{A}_3$ that transforms the origin in the point $(3,1,1)$ and whose plane π of cartesian equation $x_1+2x_2-x_3+1=0$, is a plane of fixed points.

Solution.

As the plane π is a plane of fixed points, the vector plane associated with π is a plane of eigenvectors associated with the eigenvalue $\lambda=1$ of the associated linear transformation $\bar{f}$.

As $\pi \equiv P+\langle \overline{u}_1, \overline{u}_2 \rangle$ with $P(0,0,1)$, $\overline{u}_1=(1,0,1)$, $\overline{u}_2=(0,1,2)$ then $P\in\pi$ (this is, the coordinates of P are a solution of the equation of π) and the vectors $\overline{u}_1$,$\overline{u}_2 \in \overline{\pi}$ (their coordinates are a solution of the associated homogeneous equation: $x_1+2x_2-x_3=0$).

Therefore, we have:

$$\begin{aligned} f(0,0,0) &= (3,1,1) \\ f(0,0,1) &= (0,0,1) \\ \bar{f}(\overline{u}_1) &= \overline{u}_1 \Longrightarrow \bar{f}(1,0,1) = (1,0,1) \\ \bar{f}(\overline{u}_2) &= \overline{u}_2 \Longrightarrow \bar{f}(0,1,2) = (0,1,2) \end{aligned}$$

From the first two conditions we obtain

$$\begin{aligned} \bar{f}(\overline{OP}) &= f(P) - f(O) = (0,0,1) - (3,1,1) \\ &= (-3,-1,0). \end{aligned}$$

Thus, considering the coordinate system $\mathcal{R}' = \left\{P; \overline{OP}, \overline{u}_1, \overline{u}_2\right\}$ (notice that $\overline{OP}$, $\overline{u}_1$, $\overline{u}_2$ are linearly independent), we obtain:

$$M_f(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

Since

$$M\left(\mathcal{R}'\mathcal{R}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

we have

$$M_f(\mathcal{R},\mathcal{R}) = M_f(\mathcal{R}',\mathcal{R}) \cdot M(\mathcal{R}',\mathcal{R})^{-1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 4 & 6 & -3 \\ 2 & 1 & 3 & -1 \\ 1 & 1 & 2 & 0 \end{pmatrix}.$$

So the analytic expression of f is:

$$f(x_1, x_2, x_3) = (6 + 4x_1 + 6x_2 - 3x_3, 2 + x_1 + 3x_2 - x_3, 1 + x_1 + 2x_2).$$