CHAPTER II: AFFINE AND EUCLIDEAN GEOMETRY

4. ISOMETRIES

Definition

Let $(\mathbb{E}, E, \phi)$ and $(\mathbb{E}', E', \phi')$ be two euclidean affine spaces. An affine transformation $f : \mathbb{E} \longrightarrow \mathbb{E}'$ is an *isometry* if

$$d'\left(f(P), f(Q)\right) = d(P, Q), \quad \forall P, Q \in \mathbb{E},$$

where d is the distance defined in $\mathbb{E}$ and d' is the distance defined in $\mathbb{E}'$.

Observation Isometries are always injective, because if $f(P) = f(Q)$ then

$$0 = d\left(f(P), f(Q)\right) = d(P, Q)$$

implies $P = Q$.

Proposition

An affine transformation $f\colon \mathbb{E} \longrightarrow \mathbb{E}'$ is an isometry if and only if its associated linear transformation $\bar{f}\colon E \longrightarrow E'$ preserves the dot product (this is, $\bar{f}$ is a vector isometry).
Vector isometries are also called orthogonal endomorphisms.

Some properties of vector isometries Let $\overline{f} : E \longrightarrow E'$ be a vector isometry.

1. For all $\overline{u} \in E$ then $\|\overline{u}\| = \|f(\overline{u})\|$.
2. For all $\overline{u}, \overline{v} \in E$ then
$$\cos(\widehat{\overline{u}, \overline{v}}) = \frac{\overline{u} \cdot \overline{v}}{\|\overline{u}\| \, \|\overline{v}\|} = \frac{\bar{f}(\overline{u}) \cdot \bar{f}(\overline{v})}{\left\|\bar{f}(\overline{u})\right\| \, \left\|\bar{f}(\overline{v})\right\|} = \cos(\widehat{\bar{f}(\overline{u}), \bar{f}(\overline{v})}).$$
3. If λ is a real eigenvalue of $\overline{f}$ then $\lambda = \pm 1$.

4. If 1 and -1 are eigenvalues of $\overline{f}$ then the eigenspaces V_1 and V_{-1} are orthogonal subspaces.

5. Let $E = E'$ with dimension n, the matrix A associated to $\overline{f}$ in a orthonormal basis of E is an orthogonal matrix, that is $A^{-1} = A^t$ or equivalently $AA^t = I_n$.

Definition

A *movement* is an isometry of an euclidean affine space $\mathbb{E}$ on itself.

4.1 Classification of isometries

The linear transformation $\overline{f}$ associated to a movement $f\colon \mathbb{E} \longrightarrow \mathbb{E}$, is orthogonal, therefore, in an orthonormal coordinate system $\mathcal{R} = \{O, B\}$; the matrix associated to f has the form:

$$M_f(R) = \begin{pmatrix} 1 & \overline{0}^t \\ \overline{Of(O)} & A \end{pmatrix}$$

where $A = M_B(\bar{f})$ is an orthogonal matrix; this is, $A^{-1} = A^t$. Therefore, $\det A = \pm 1$.

If $\det A = 1$ we say that the isometry is a *direct* isometry.

If $\det A = -1$ we say that the isometry is an *indirect* isometry.

4.1.1 Isometries in the affine euclidean plane

Let f be an isometry of an euclidean affine space $\mathbb{E}$ of dimension 2 on itself. Let $\mathcal{R} = \{O, B = (\overline{e}_1, \overline{e}_2)\}$ be an orthonotmal coordinate system in $\mathbb{E}$. The matrix associated to f with respect to $\mathcal{R}$ is

$$M_f(R) = \begin{pmatrix} 1 & \overline{0}^t \\ \overline{b} & A \end{pmatrix} \text{ with } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The charasteristic polynomial of A is $\det(A - \lambda I) = \lambda^2 - tr(A)\lambda + \det(A)$.

Subspace of fixed points

The equation of the subspace of fixed points of f is

$$(A - I)X + \bar{b} = \bar{0}.$$

Therefore, f has fixed points if the former equation has a solution.

If $\operatorname{rank}(A - I) = 2$ (then also $\operatorname{rank}(A - I|\bar{b}) = 2$) then f has only one fixed point.

If $\operatorname{rank}(A - I) = \operatorname{rank}(A - I|\bar{b}) = 1$ then f has a line of fixed points.

If $\operatorname{rank}(A - I) = \operatorname{rank}(A - I|\bar{b}) = 0$ then f is the identity transformation.

1. If $\det A = 1$, the isometry f is direct and $A \in SO(2)$ (matrices of order 2, orthogonal and with determinant 1). There exists an angle θ such that

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Notice that, in this case, $\det(A-\lambda I) = \lambda^2 - tr(A)\lambda + 1$ and $tr(A) = 2\cos\theta$.

a) If $\cos\theta = \frac{1}{2}tr(A) \neq 1$, then $\lambda = 1$ is not an eigenvalue of the matrix A and, therefore $\operatorname{rank}(A-I) = 2$ and f has only one fixed point that we denote by P. In this case, f is a *rotation of angle* θ *and with center in the fixed point* P. In the coordinate system $\mathcal{R}' = \{P, B = (\overline{e}_1, \overline{e}_2)\}$ the matrix associated to f is

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

If $\cos\theta = \frac{1}{2}tr(A) = -1$ then $\theta = 180^o$ and f is a *central symmetry with center in the fixed point* P.

b) If $\cos\theta = \frac{1}{2}tr(A) = 1$, then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and f is a *translation of vector* $\bar{b}$.

1) $\mathrm{rank}(A - I) = \mathrm{rank}(A - I|\bar{b}) = 0$ then f is the *identity transformation*.

2) $\mathrm{rank}(A - I) \neq \mathrm{rank}(A - I|\bar{b})$ then f is the *translation of vector* $\bar{b}$.

2. If $\det(A) = -1$ the isometry f is indirect and $A \in O(2)$ (orthogonal matrices of order 2). The eigenvalues of A are $1, -1$. If we take $\overline{u}_1$ eigenvector associated to 1 and $\overline{u}_2$ eigenvector associated to -1, we have that in the basis $B' = \{\overline{u}_1, \overline{u}_2\}$ the matrix associated to $\bar{f}$ (which we keep on calling A) is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $\operatorname{rank}(A - I) = 1$.

a) If $\operatorname{rank}(A - I|\overline{b}) = 1$ then there exists a line of fixed points of f. Let P be a point of this line (this is, a fixed point of f), in the orthonormal coordinate system $\mathcal{R}' = \left\{P, \left\{\frac{1}{\|\overline{u}_1\|}\overline{u}_1, \frac{1}{\|\overline{u}_2\|}\overline{u}_2\right\}\right\}$ the matrix associated to f is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and f is an *axial symmetry*. The line of fixed points $r \equiv P + \langle \overline{u}_1 \rangle$ is called *axis of the symmetry*.

b) If $\operatorname{rank}(A - I|\overline{b}) = 2$ then f has no fixed points. In the orthonormal coordinate system $\mathcal{R}' = \left\{O, \left\{\frac{1}{\|\overline{u}_1\|}\overline{u}_1, \frac{1}{\|\overline{u}_2\|}\overline{u}_2\right\}\right\}$ the matrix associated to f is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \\ c_2 & 0 & -1 \end{pmatrix}.$$

Let us study whether in this case there exists any invariant line. We know that $V(1) = \langle \overline{u}_1 \rangle$ and $V(-1) = \langle \overline{u}_2 \rangle$. Let us compute $\overline{Xf(X)}$. Let (x_1', x_2') be the coordinates in the coordinate system $\mathcal{R}'$ of an arbitrary point X, we have:

$$\begin{aligned} \overline{Xf(X)} &= f(X) - X = (x_1' + c_1, -x_2' + c_2) - (x_1', x_2') \\ &= (c_1, -2x_2' + c_2). \end{aligned}$$

If $-2x'_2+c_2=0$ then $\overline{Xf(X)} \in \langle \overline{u}_1 \rangle$. Therefore, the line with equation $-2x'_2+c_2=0$ is an invariant line of f. If we take a point P as the origin of the coordinate system in the above mentioned line (then the coordinates of P have the form $(p, \frac{c_2}{2})$), we have that in the coordinate system $\mathcal{R}' = \left\{ P, \left\{ \frac{1}{\|\overline{u}_1\|}\overline{u}_1, \frac{1}{\|\overline{u}_2\|}\overline{u}_2 \right\} \right\}$ the matrix of f is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thi is the *composition of an axial symmetry, with axis the invariant line* $P+\langle \overline{u}_1 \rangle$ *, and a translation parallel with the axis (with vector* $(p,0)$*).*

Observation. Every symmetry composed with a translation can be decomposed in an unique way as a symmetry composed with a translation with the direction vector of the axis as its vector.

Table of classification

$\det A = 1$ (then $\cos\alpha = \frac{1}{2} trA$)

	$rg(A-I)$	$rg(A-I\,\vert\,\bar{b})$	Classification
$\cos\alpha = 1$	0	$0\ (\bar{b}=\bar{0})$	Identity
$\cos\alpha = 1$	0	$1\ (\bar{b}\neq\bar{0})$	Translation
$\cos\alpha \neq 1$	2	2	Rotation

$\det A = -1$

$rg(A-I)$	$rg(A-I\,\vert\,\bar{b})$	Classification
1	1	Symmetry with respect to line of fixed points
1	2	Translational symmetry

Example

Classify the isometry $f(x_1, x_2) = (1 - x_2, 3 - x_1)$.

Solution

The matrix associated with this isometry is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}, \text{ we denote by } A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

As $\det(A) = -1$ the isometry is indirect, has eigenvalues $\lambda = -1, 1$ and, in this case, $\overline{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is an unitary eigenvector associated to the eigenvalue $\lambda = -1$ and $\overline{e}_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is an unit eigenvector associated with the eigenvalue $\lambda = 1$. Let us see if f has fixed points. As

$$\operatorname{rank}(A - I) = 1 \text{ and } \operatorname{rank}(A - I|\overline{b}) = 2$$

the isometry f has no fixed points.

It is an isometry composed with a translation. Let us check if it has any invariant line, and let us calculate it

$$\begin{aligned} \overline{Xf(X)} &= f(X) - X = (1 - x_2, 3 - x_1) - (x_1, x_2) \\ &= (1 - x_1 - x_2, 3 - x_1 - x_2) \in V(-1) \text{ or } V(1) \end{aligned}$$

Then, $\overline{Xf(X)} \in V(-1)$ if and only if $\overline{Xf(X)}$ and $\overline{e}_1$ are proportional; this is, if

$$\begin{aligned} 1 - x_1 - x_2 &= t \\ 3 - x_1 - x_2 &= t \end{aligned}$$

Subtracting both equations we obtain $2 = 0$ which is impossible.
And $\overline{Xf(X)} \in V(1)$ if and only if $\overline{Xf(X)}$ and $\overline{e}_2$ are proportional; this is, if

$$\begin{aligned} 1 - x_1 - x_2 &= -t \\ 3 - x_1 - x_2 &= t \end{aligned}$$

Subtracting both equations we obtain $t = 1$ and therefore, $\overline{Xf(X)} \in V(1)$ if and only if

$$x_1 + x_2 = 2$$

which is the equation of the invariant line.
Therefore, f is a translational symmetry; this is, a symmetry s with the invariant line as axis, composed with a translation with vector proportional to the eigenvector associated to the eigenvalue $\lambda = 1$ (direction vector of the invariant line). The matrix of the symmetry is

$$M_s(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & -1 \\ b & -1 & 0 \end{pmatrix}$$

where a, b let us fix any point of the line $x_1 + x_2 = 2$. For example, let us impose it fixes the point $(1, 1)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 0 & -1 \\ b & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{cases} a = 2 \\ b = 2 \end{cases}$$

Let us compute which is the translation vector:

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ v_1+2 & 0 & -1 \\ v_2+2 & -1 & 0 \end{pmatrix}$$

then $v_1 = -1$ and $v_2 = 1$.

Example

Obtain the analytic expression of the isometry of the plane which is a composition of the symmetry with the line $x_1 + x_2 = 1$ as axis and the translation with vector $\overline{v} = (1, 2)$. Decompose the obtained isometry as composition of a symmetry and a translation with vector parallel with the symmetry axis.

Solution

The vector line associated to the symmetry axis has cartesian equation $x_1 + x_2 = 0$.

Let us consider the coordinate system $\mathcal{R}' = \{P, \{\overline{u}_1, \overline{u}_2\}\}$ where P is a point of the symmetry axis, for example, $P(1, 0)$, vector $\overline{u}_1$ is an unitary vector in the line $x_1 + x_2 = 0$; for example $\overline{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and vector $\overline{u}_2$ is an unitary vector and orthogonal to $\overline{u}_1$; this is, $\overline{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

In the mentioned coordinate system the matrix associated to the symmetry S with axis $x_1 + x_2 = 1$ is

$$M_S(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} M_S(\mathcal{R}) &= M\left(\mathcal{R}'\mathcal{R}\right) M_S\left(\mathcal{R}'\right) M\left(\mathcal{R}\mathcal{R}'\right) \\ &= M\left(\mathcal{R}'\mathcal{R}\right) M_S(\mathcal{R}')(M\left(\mathcal{R}'\mathcal{R}\right))^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \end{aligned}$$

The translation T with vector $\overline{v} = (1, 2)$ has associated matrix:

$$M_T(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Thus, the matrix associated to the isometry we are looking for, is

$$\begin{aligned} M_{T\circ S}(\mathcal{R}) &= M_T(\mathcal{R}) M_S(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}. \end{aligned}$$

And

$$(T \circ S)(x_1, x_2) = (2 - x_2, 3 - x_1).$$

We are going to decompose the obtained isometry as a composition of a symmetry and a translation t_2 with vector parallel with the symmetry axis. Let us decompose the vector $\overline{v} = (1,2)$ as an addition of a vector with direction parallel with the symmetry axis s and a vector orthogonal to this forementioned vector:

$$\overline{v} = (1,2) = a(1,-1) + b(1,1),$$

from where $a = -\frac{1}{2}$ and $b = \frac{3}{2}$. Therefore, let us take the translation t_2 with vector $\overline{v}_2 = (-\frac{1}{2}, \frac{1}{2})$. Let us calculate the symmetry s_2:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ c & 0 & -1 \\ d & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ c-\frac{1}{2} & 0 & -1 \\ d+\frac{1}{2} & -1 & 0 \end{pmatrix}$$

from where, $c = \frac{5}{2}$ and $d = \frac{5}{2}$. Then,

$$M_{s_2}(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 0 & -1 \\ \frac{5}{2} & -1 & 0 \end{pmatrix}.$$

Now let us compute the line of fixed points of the symmetry s_2. We have:

$$\begin{aligned} \overrightarrow{Xs_2(X)} &= \left(\frac{5}{2} - y,\ \frac{5}{2} - x\right) - (x,\ y) \\ &= \left(\frac{5}{2} - x - y,\ \frac{5}{2} - x - y\right) \\ &= \left(\frac{5}{2} - x - y\right)(1, 1). \end{aligned}$$

Therefore, the line $5 = 2x + 2y$ is the line of fixed points of the symmetry s_2 (is the symmetry axis).

4.2 Isometries in the tridimensional euclidean affine space

Let f be an isometry of an euclidean affine space $\mathbb{E}$ of dimension 3 on itself. Let $\mathcal{R} = \{O, B = (\bar{e}_1, \bar{e}_2, \bar{e}_3)\}$ be an orthonormal coordinate system in $\mathbb{E}$. The matrix associated to f with respect to $\mathcal{R}$ is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & \bar{0}^t \\ \bar{b} & A \end{pmatrix} \text{ with } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } \bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The characteristic polynomial of A is $\det(A - \lambda I) = -\lambda^3 + tr_2(A)\lambda^2 - tr(A)\lambda + \det(A)$, where

$$tr_2(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

Subspace of fixed points

The equation of the subspace of fixed points of f is

$$(A - I)X + \overline{b} = \overline{0}.$$

Therefore, f has fixed points if the above mentioned equation has any solution.
If $\operatorname{rank}(A - I) = 3$ (then also $\operatorname{rank}(A - I|\overline{b}) = 3$) then f has only one fixed point.

If $\operatorname{rank}(A - I) = \operatorname{rank}(A - I|\overline{b}) = 2$ then f has a line of fixed points.

If $\operatorname{rank}(A - I) = \operatorname{rank}(A - I|\overline{b}) = 1$ then f has a plane of fixed points.

If $\operatorname{rank}(A - I) = \operatorname{rank}(A - I|\overline{b}) = 0$ then f is the identity transformation.

1. If $\det A = 1$, the isometry f is direct and $A \in SO(3)$ (orthogonal matrices of order 3 and with determinant 1) and, in a convenient orthonormal basis $B' = \{\overline{u_1}, \overline{u_2}, \overline{u_3}\}$ the matrix associated to $\bar{f}$ is written:

$$M_{\overline{f}}(B') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Notice that, in this case, $tr(A) = 1 + 2\cos\theta$.

a) If $\cos\theta = 1$, then $\operatorname{rank}(A - I) = 0$, then we can encounter two situations:

1) $\operatorname{rank}(A - I|\bar{b}) = 0$ and, in this case,

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and f is the *identity transformation*.

2) $\mathrm{rank}(A - I|\bar{b}) = 1$ and, in this case, there are no fixed points and f is a *translation with vector* $\bar{b}$. The matrix associated with f in this case is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix}.$$

b) If $|\cos\theta| \neq 1$, then $rank(A-I)=2$ and we can encounter two situations:

1) $rank(A-I|\overline{b})=2$ and, in this case, there is a line of fixed points $r \equiv Q+\langle\overline{w}_1\rangle$, where $\overline{w}_1$ is an eigenvector associated with the eigenvalue $\lambda=1$. In the coordinate system

$$\mathcal{R}' = \left\{Q, \left\{\overline{u}_1 = \frac{1}{\|\overline{w}_1\|}\overline{w}_1, \overline{u}_2, \overline{u}_3\right\}\right\}$$

the matrix associated to f is

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Thus f is a *rotation of angle θ and axis the line r of fixed points.*

In the particular case where $\cos\theta=-1$, we would have an *axial symmetry with the line r of fixed points as axis.*

2) $\mathrm{rank}(A - I|\overline{b}) = 3$ and, in this case, there are no fixed points. The matrix associated with f can be written as follows:

$$
\begin{aligned}
M_f(\mathcal{R}') &= \begin{pmatrix} 1 & \overline{0}^t \\ \overline{b} & A \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}
\end{aligned}
$$

and f is an *helical movement,* this is, a rotation of angle θ and axis the invariant line f, with associated vector subspace $V(1)$, composed with a translation parallel with the fore mentioned line (with vector $\overline{u} = \overline{Xf(X)}$, where $X \in r$).

2. If $\det A = -1$, the isometry f is indirect and $A \in O(3)$ (orthogonal matrices of order 3).

   In a convenient orthonormal basis $B' = \{\overline{u}_1, \overline{u}_2, \overline{u}_3\}$,the unitary vector $\overline{u}_1$ is the eigenvector associated with $\lambda = -1$, the matrix associated with $\bar{f}$ is written as follows:

$$M_{\overline{f}}(B') = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

   Notice that, in this case, $tr(A) = -1 + 2\cos\theta$.

   a) If $\cos\theta = 1$ then $\operatorname{rank}(A - I) = 1$.

   1) If $\operatorname{rank}(A - I|\overline{b}) = 1$ then there exists a plane of fixed points $\pi \equiv P + \langle \overline{v}_1, \overline{v}_2 \rangle$. In the coordinate system

$$\mathcal{R}' = \left\{Q, \left\{\overline{u}_1, \overline{u}_2 = \frac{1}{\|\overline{v}_1\|}\overline{v}_1, \overline{u}_3 = \frac{1}{\|\overline{v}_2\|}\overline{v}_2\right\}\right\}$$

the matrix associated to f is written as follows

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and f is a *specular symmetry (or reflection) with respect to the plane of fixed points.*

2) If $\mathrm{rank}(A - I|\overline{b}) = 2$ then there are no fixed points. The matrix associated with f can be written as follows:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & \overline{0}^t \\ \overline{b} & A \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and f is a *symmetry composed with a translation with vector parallel with the invariant plane* ($\overline{v} = (0, c_2, c_3)$).

b) If $\cos\theta \neq 1$ then $\bar{f}$ does not have the eigenvalue $\lambda = 1$ and there is an unique fixed point Q. In the orthonormal coordinate system $\mathcal{R}' = \{Q, \{\overline{u}_1, \overline{u}_2, \overline{u}_3\}\}$ the matrix associated with f is written:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and f is a *symmetry (with respect to the plane* $Q + \langle \overline{u}_2, \overline{u}_3 \rangle$*) composed with a rotation of angle* θ *and axis* $Q + \langle \overline{u}_1 \rangle$.

In the particular case where $\cos\theta = -1$, then f is a *central symmetry with the only fixed point* Q *as the center*.

Table of classification

$$\det A = 1$$

$$\cos(\alpha) = \frac{1}{2}(trA - 1)$$

$rg(A-I)$	$rg(\bar{b}\,|\,A-I)$	Classification
0	$0 \ (\bar{b} = \bar{0})$	Identity
0	$1 \ (\bar{b} \neq \bar{0})$	Translation
2	2	Rotation of angle α
2	3	Helical movement

$$\det A = -1$$

$$\cos(\alpha) = \frac{1}{2}(trA + 1)$$

$rg(A-I)$	$rg(\bar{b} \mid A-I)$	Classification
1	1	Symmetry
1	2	Translational symmetry
3	3	Composition of a rotation and a symmetry