CHAPTER III: CONICS AND QUADRICS

I freely confess that I never had a taste for study or research either in physics or geometry except in so far as they could serve as a means of arriving at some sort of knowledge of the proximate causes...for the good and convenience of life, in maintaining health, in the practice of some art...having observed that a good part of the arts is based on geometry, among others the cutting of stone in architecture, that of sundials, and that of perspective in particular.

Gèrard Desargues (1591-1661)

1. INTRODUCTION TO THE PROJECTIVE SPACE

1.1 Definitions

Let V_{n+1} be an $(n+1)$-dimensional vector space. The projective space of dimension n over V_{n+1} is the set of all vector lines of V_{n+1}. It is denoted by

$$\mathbb{P}_n(V_{n+1}) = \{\langle v \rangle \mid v \in V_{n+1} - \{\overline{0}\}\}.$$

Every vector in V_{n+1} determines a projective point.

Examples

We call the set of vector lines of $\mathbb{R}^3$ real projective plane and we denote it by $\mathbb{P}_2$; this is

$$\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3) = \{< \overline{v} > \ \mid \overline{v} \in \mathbb{R}^3 - \{(0,0,0)\}\}.$$

We call the set of vector lines of $\mathbb{R}^4$ real projective space and denote it by $\mathbb{P}_3$; this is

$$\mathbb{P}_3 = \mathbb{P}(\mathbb{R}^4) = \{< \overline{v} > \ \mid \overline{v} \in \mathbb{R}^4 - \{(0,0,0,0)\}\}.$$

1.2 Homogeneous coordinates

Let $\mathbb{P}_n(V_{n+1})$ be a projective space. We say that a family of points $\{< \overline{v}_1 >, ..., < \overline{v}_r >\}$ of $\mathbb{P}_n(V_{n+1})$ generate the projective space $\mathbb{P}_n(V_{n+1})$ if the family of vectors $\{\overline{v}_1, ..., \overline{v}_r\}$ generates the vector space V_{n+1}.

Let $\mathbb{P}_n(V_{n+1})$ be a projective space. We say that the points $< \overline{v}_1 >, ..., < \overline{v}_r >$ of $\mathbb{P}_n(V_{n+1})$ are projectively independent if the vectors $\overline{v}_1, ..., \overline{v}_r$ of V_{n+1} are linearly independent.

Example

Let us consider $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{R}^3)$, then an independent generating family of points of $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{R}^3)$ is formed by three points $X_1 =< \overline{v}_1 >$, $X_2 =< \overline{v}_2 >$ and $X_3 =< \overline{v}_3 >$ so that the three vectors $\overline{v}_1, \overline{v}_2$ and $\overline{v}_3$ are linearly independent. A point $X =< \overline{w} >\in \mathbb{P}_2$ can be expressed as follows:

$$\overline{w} = \alpha_1\overline{v}_1 + \alpha_2\overline{v}_2 + \alpha_3\overline{v}_3,$$

and the coordinates of X would be $(\alpha_1, \alpha_2, \alpha_3)$.

If we choose the representative $\lambda\overline{w}$, $\lambda \neq 0$ of X, as $X =< \lambda\overline{w} >\in \mathbb{P}_2$ then

$$\lambda\overline{w} = \lambda\alpha_1\overline{v}_1 + \lambda\alpha_2\overline{v}_2 + \lambda\alpha_3\overline{v}_3,$$

and the coordinates of X would be $(\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3)$.

We call the class $[\alpha_1, \alpha_2, \alpha_3]$ homogeneous coordinates of the projective point X; this is,

$$[\alpha_1, \alpha_2, \alpha_3] = \{(\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3), \text{ with } \lambda \neq 0\}.$$

1.3 Relationship between affine space and projective space

Let $\mathbb{A}_n$ be an affine space with associated vector space $\mathbb{R}^n$.

Let us consider a coordinate system $\mathcal{R} = \{O, B\}$ of $\mathbb{A}_n$.
Given $X \in \mathbb{A}_n$ with cartesian coordinates $(x_1, ..., x_n)$ then

$$(\lambda, \lambda x_1, \ldots, \lambda x_n) \text{ with } \lambda \neq 0$$

is a set of homogeneous coordinates of X. We choose $(1, x_1, \ldots, x_n)$ as representative of the homogeneous coordinates of X.

Definition Given an affine line $P + \langle v \rangle$ were $P \in \mathbb{A}_n$ and $v \in \mathbb{R}^n$ with coordinates $(v_1, \ldots, v_n)$ in the basis B then we call $(0, v_1, \ldots, v_n)$ the point at infinity of the affine line.

Definition. Let $\mathbb{A}_n$ be an affine space with associated vector space $\mathbb{R}^n$ with coordinate system $\mathcal{R} = \{O, B\}$. We call the set formed by all the points of $\mathbb{A}_n$ and the points at infinity of $\mathbb{A}_n$ *projectivized affine space* and denote it by $\overline{\mathbb{A}}_n$; this is

$$\overline{\mathbb{A}}_n = \mathbb{A}_n \cup \{(0, x_1, x_2, ..., x_n) \mid x_i \in \mathbb{R}\}.$$

We identify $\overline{\mathbb{A}}_n$ with $\mathbb{P}_n(\mathbb{R}^{n+1})$ in the following way:

$$\overline{\mathbb{A}}_n \longleftrightarrow \mathbb{P}_n(\mathbb{R}^{n+1})$$

$$(1, \frac{x_1}{x_0}, ..., \frac{x_n}{x_0}) \longrightarrow \langle (x_0, x_1, ..., x_n) \rangle, \quad (x_0 \neq 0) \text{ proper points of } \mathbb{P}(\mathbb{R}^{n+1})$$

$$(0, x_1, ..., x_n) \longrightarrow \langle (0, x_1, ..., x_n) \rangle, \quad (x_0 = 0) \text{ improper points of } \mathbb{P}(\mathbb{R}^{n+1})$$

1.4 Equations of the lines of the projective plane

Let $\mathbb{P}_2$ be the real projective plane.

Given two independent points $P, Q \in \mathbb{P}_2$, we have $P =< \overline{v} >$ and $Q =< \overline{w} >$ with $\overline{v}$,$\overline{w} \in \mathbb{R}^3$ linearly independent vectors, the line r that contains P and Q is

$$r = \{< \lambda\overline{v} + \mu\overline{w} > \ \mid (\lambda, \mu) \neq (0,0)\}.$$

If the points P and Q have the following homogeneous coordinates:

$$P = [p_0, p_1, p_2], \ \ Q = [q_0, q_1, q_2]$$

then a point $X \in r$ if and only if its coordinates $[x_0, x_1, x_2]$ verify the following equations

$$\begin{cases} \alpha x_0 = \lambda p_0 + \mu q_0 \\ \alpha x_1 = \lambda p_1 + \mu q_1 \\ \alpha x_2 = \lambda p_2 + \mu q_2 \end{cases}, \ \ (\alpha, \lambda, \mu) \neq (0,0,0),$$

which are called *parametric equations* of the line r of the projective plane $\mathbb{P}_2$.

Equivalently the point $X = [x_0, x_1, x_2] \in r$ if and only if

$$a_0x_0 + a_1x_1 + a_2x_2 = 0,$$

which is the *cartesian equation* of the line that is obtained when we demand the following determinant to be zero:

$$0 = \begin{vmatrix} x_0 & p_0 & q_0 \\ x_1 & p_1 & q_1 \\ x_2 & p_2 & q_2 \end{vmatrix}.$$

1.4.1 Relationship between the lines of the real affine plane and the projective plane.

Let $\mathbb{A}_2$ be the affine plane with coordinate system $\mathcal{R} = \{O, B\}$ and let us consider the line r of the affine plane $\mathbb{A}_2$ with equation $a_0 + a_1x_1 + a_2x_2 = 0$.

Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two points of the line, then two points of the projective plane $[1, p_1, p_2]$, $[1, q_1, q_2]$ determine a line r of the projective plane $\mathbb{P}_2$ with equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$, which is called *line of* $\mathbb{P}_2$ *associated to the affine line* r.

Reciprocally, given a line r of the projective plane $\mathbb{P}_2$ with equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$. If $p_0 \neq 0$, then the point of the affine plane $\left(\frac{p_1}{p_0}, \frac{p_2}{p_0}\right)$ is in the line r of the affine plane $\mathbb{A}_2$ with equation:

$$a_0 + a_1x_1 + a_2x_2 = 0.$$

Definition. The line that joins two proper points of $\mathbb{P}_2$ is called *a proper line* of $\mathbb{P}_2$.

Every proper line $a_0x_0 + a_1x_1 + a_2x_2 = 0$, determines a point at infinity $[0, -a_2, a_1]$ where $(-a_2, a_1)$ is the direction vector of the line r of the affine plane $\mathbb{A}_2$ with equation $a_0 + a_1x_1 + a_2x_2 = 0$.

Definition. The line that joins two points at infinity of $\mathbb{P}_2$ is called *infinity or improper line* of $\mathbb{P}_2$ and has equation $x_0 = 0$.

1.5 Equations of projective subspaces of $\mathbb{P}_3$

Let $\mathbb{P}_3$ be the real tridimensional projective space.

1.5.1 Lines in $\mathbb{P}_3$

Let P, Q be two independent points of $\mathbb{P}_3$. Therefore, $P =< \overline{v} >$ and $Q =< \overline{w} >$ with $\overline{v}, \overline{w} \in \mathbb{R}^4$ linearly independent vectors. The line r that contains P and Q is

$$r = \{< \lambda\overline{v} + \mu\overline{w} > \ \mid (\lambda, \mu) \neq (0,0)\}.$$

If the points P and Q have the following homogeneous coordinates:

$$P = [p_0, p_1, p_2, p_3], \quad Q = [q_0, q_1, q_2, q_3]$$

then a point $X = [x_0, x_1, x_2, x_3] \in r$ if and only if its coordinates verify the following equations

$$\left\{ \begin{array}{l} \alpha x_0 = \lambda p_0 + \mu q_0 \\ \alpha x_1 = \lambda p_1 + \mu q_1 \\ \alpha x_2 = \lambda p_2 + \mu q_2 \\ \alpha x_3 = \lambda p_3 + \mu q_3 \end{array} \right. , \ (\alpha, \lambda, \mu) \neq (0, 0, 0),$$

which are called *parametric equations* of the line r of the projective space $\mathbb{P}_3$.

Equivalently the point $X = [x_0, x_1, x_2, x_3]$ belongs to the line r of the projective space $\mathbb{P}_3$ if and only if

$$\operatorname{rank}\begin{pmatrix} x_0 & p_0 & q_0 \\ x_1 & p_1 & q_1 \\ x_2 & p_2 & q_2 \\ x_3 & p_3 & q_3 \end{pmatrix} = 2,$$

from where we obtain the two *cartesian equations* of the line.

Definition. The line that joins two proper points of $\mathbb{P}_3$ is called *a proper line* of $\mathbb{P}_3$. Its equations are the homogeneous equations of an affine line.

Definition. The line that joins two improper points of $\mathbb{P}_3$ is called *improper or infinity line* of $\mathbb{P}_3$.

Observation. In $\mathbb{P}_3$ there is an infinite number of improper lines.

1.5.2 Planes in $\mathbb{P}_3$

Given three independent points $P =< \overline{v} >$, $Q =< \overline{w} >$ and $R =< \overline{u} >$ of $\mathbb{P}_3$, the plane that contains P, Q and R is

$$\pi = \{< \lambda\overline{v} + \mu\overline{w} + \gamma\overline{u} > \ \mid (\lambda, \mu, \gamma) \neq (0, 0, 0)\}.$$

If the points P, Q and R have the following homogeneous coordinates:

$$P = [p_0, p_1, p_2, p_3]$$
$$Q = [q_0, q_1, q_2, q_3]$$
$$R = [r_0, r_1, r_2, r_3]$$

then a point $X = [x_0, x_1, x_2, x_3]$ belongs to the plane π of the projective space $\mathbb{P}_3$ if and only if its coordinates verify the following equations

$$\begin{cases} \alpha x_0 = \lambda p_0 + \mu q_0 + \gamma r_0 \\ \alpha x_1 = \lambda p_1 + \mu q_1 + \gamma r_1 \\ \alpha x_2 = \lambda p_2 + \mu q_2 + \gamma r_2 \\ \alpha x_3 = \lambda p_3 + \mu q_3 + \gamma r_3 \end{cases}, \ (\alpha, \lambda, \mu, \gamma) \neq (0, 0, 0, 0)$$

which are called *parametric equations* of the plane π of the projective space $\mathbb{P}_3$.

Equivalently the point $X = [x_0, x_1, x_2, x_3]$ is contained in the plane π of the projective space $\mathbb{P}_3$ if and only if

$$a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

which is the *cartesian equation* of the plane that is obtained when we force that the following determinant is to be zero:

$$0 = \begin{vmatrix} x_0 & p_0 & q_0 & r_0 \\ x_1 & p_1 & q_1 & r_1 \\ x_2 & p_2 & q_2 & r_2 \\ x_3 & p_3 & q_3 & r_3 \end{vmatrix}.$$

Observations.

Three proper points determine a *proper plane* of $\mathbb{P}_3$. Its equation is the homogeneous equation of an affine plane.

Three improper points determine an *improper plane* of $\mathbb{P}_3$ which has as cartesian equation the equation $x_0 = 0$.

Every proper plane determines a line at infinity. Every line at infinity is contained in the infinity plane $x_0 = 0$.