CHAPTER III: CONICS AND QUADRICS

2. CONICS

2.5 Affine classification and notable elements of conics

Let $\overline{\mathbb{A}}_2 = \mathbb{P}_2(\mathbb{R}^3)$ be the projectivized affine plane, with coordinate system $\mathcal{R} = \{O, B\}$. Let ω be a quadratic form with associated matrix A. Let

$$\bar{C} = \{X \in \mathbb{P}_2(\mathbb{R}^3) \mid \omega(X) = 0\}$$

be a projective conic with affine conic

$$C = \bar{C} \cap \mathbb{A}_2 = \{X \in \mathbb{A}_2 \mid \omega(\tilde{X}) = 0\}, \text{ where } \tilde{X} = (1, x_1, x_2).$$

2.5.1 Center of an affine conic

Definition: We call *center* of an affine conic C the pole of the infinity line if that point is a proper point (if it is not, we say that the affine conic does not have a proper center).

The equation of the infinity line is $x_0 = 0$ and the equation of the conic is $X^t A X = 0$. Therefore, the pole of the infinity line is the point P such that $P^t A = (1, 0, 0)$.

Example. The parabola is tangent to the infinite: therefore, the pole if the infinity line is the tangent point, which is at infinity, so the parabola has no proper center.

Proposition. The center of an affine conic is its center of symmetry.

2.6 Relative position between the conic and the line at infinity

1. If the line at infinity $r_\infty \equiv x_0 = 0$ is not tangent to the conic $\bar{C}$ then the pole of r_∞ is a proper point; $\bar{C}$ has a center that we denote by C and the coordinates of this center are

$$Z = [c_0, c_1, c_2] \text{ such that } (c_0, c_1, c_2)A = (1, 0, 0).$$

2. If the infinity line $r_\infty \equiv x_0 = 0$ is tangent to the conic $\bar{C}$ then the pole of r_∞, if it exists, is the tangent point. In such case,

$$\bar{C} \cap r_\infty = \{\text{center}\}$$

and the center is a *double point*. If the matrix is

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

we have:

$$\bar{C} \cap r_\infty \equiv \begin{cases} a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0 \\ x_0 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 0 \\ x_0 = 0 \end{cases}$$

The second degree equation $a_{11}t^2 + 2a_{12}t + a_{22} = 0$ has discriminant:

$$\Delta_{00} = a_{12}^2 - a_{11}a_{22} = -\det(A_{00})$$

where

$$A_{00} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

- If $\det A_{00} = 0$, then $\bar{C} \cap r_\infty = \{P\}$, where P is a double point, the center at infinity of the conic.
- If $\det A_{00} \neq 0$, then $\bar{C}$ has a proper center which is the center of symmetry of the conic. Any line that contains the center intersects with the conic in two points, which are symmetric with respect to the center.

Therefore, we can encounter the following situations:

$$\bar{C} \cap r_\infty = \begin{cases} 2 \text{ different real points } (\det(A_{00}) < 0) \\ 2 \text{ imaginary conjugated points } (\det(A_{00}) > 0) \\ 1 \text{ double point } (\det(A_{00}) = 0) \end{cases}$$

2.6.1 Conics of parabollic type

Let us consider $\bar{C} \cap r_\infty = \{P\}$, P being a double point if and only if $\det A_{00} = 0$.

The center of the conic is an infinity point.

- If $\det A \neq 0$ the conic is a *parabola*.
- If $\det A = 0$ the conic is a *pair of parallel lines* $\begin{cases} \text{different if } \operatorname{rank} A = 2 \\ \text{double line if } \operatorname{rank} A = 1 \end{cases}$

2.6.2 Conics of elliptic type

Let us consider $\bar{C} \cap r_\infty = \{P_1, P_2\}$, P_1, P_2 being conjugated infinity points if and only if $\det A_{00} > 0$.
The center of the conic is a proper point.

- If $\det A \neq 0$ the conic is an *ellipse*.
- If $\det A = 0$ the conic is *a pair of infinity lines* that intersect in a real point (the singular point of the conic).

2.6.3 Conics of hyperbolic type

Let us consider $\bar{C} \cap r_\infty = \{P_1, P_2\}$, P_1, P_2 being different real points if and only if $\det A_{00} < 0$.
The center of the conic is a proper point.

- If $\det A \neq 0$ the conic is *a hyperbola*.
- If $\det A = 0$ the conic is *a pair of real distinct lines* that intersect in the singular point.

2.6.4 Notable elements of conics

Let us consider the conic $\bar{C} \equiv X^t AX = 0$, with $A^t = A$ a regular conic.

Center

We call *center* of the conic $\bar{C}$ to the pole of the infinity line (it is the center of symmetry of the conic).

Diameters and conjugated diameters

Two lines r and s that contain a point P are called *conjugated* with respect to a regular conic $\bar{C}$ when each of them contains the pole of the other.

We call *diameter of the conic* $\bar{C}$ to every line such that its pole is an infinity point.

Therefore, for each point at infinity we have a diameter.

By the fundamental Theorem of polarity, every diameter contains the pole of the line at infinity, (this is, the center), because they are polar lines of infinity points.

Asymptotes

We call *asymptote* of a conic, to a diameter which is tangent to the conic. Sometimes they do not exist. Therefore, the asymptotes are polar lines of the points at infinity of the conic.

Axes in regular conics

We will say that two lines $r' \equiv a_0x_0 + a_1x_1 + a_2x_2 = 0$ and $s' \equiv b_0x_0 + b_1x_1 + b_2x_2 = 0$ with $a_1 \neq 0$ or $a_2 \neq 0$ and $b_1 \neq 0$ or $b_2 \neq 0$ are *orthogonal* in the projective plane $\mathbb{P}_2$ if $a_1b_1 + a_2b_2 = 0$.

We call *axes of a regular conic* to those diameters that are conjugated and orthogonal at the same time.

Let us see how to obtain the axes:

Let $P[0,p_1,p_2]$ and $Q[0,q_1,q_2]$ be the points at infinity of the axes. As the axes are orthogonal lines, P and Q verify: $p_1q_1+p_2q_2=0$. On the other hand, as P and Q are the infinity points of the conjugated lines, they ought to be conjugated; this is, $P^tAQ=0$. Therefore, the following equations must hold:

$$\begin{cases} p_1q_1+p_2q_2=0 \\ (0,p_1,p_2)\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ q_1 \\ q_2 \end{pmatrix}=0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} p_1q_1+p_2q_2=0 \\ (p_1a_{11}+p_2a_{12})\,q_1+(p_1a_{12}+p_2a_{22})\,q_2=0 \end{cases}$$

$$\Longleftrightarrow \left\{ \begin{pmatrix} p_1 & p_2 \\ p_1a_{11}+p_2a_{12} & p_1a_{12}+p_2a_{22} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right.$$

This former system has a solution, different from the trivial one, if the coefficient matrix has a zero determinant; this is, if the rows of the coefficient matrix are proportional:

$$\left\{ \begin{array}{l} a_{11}p_1 + a_{12}p_2 = \lambda p_1 \\ a_{12}p_1 + a_{22}p_2 = \lambda p_2 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{c} (a_{11-}\lambda)\, p_1 + a_{12}p_2 = 0 \\ a_{12}p_1 + (a_{22} - \lambda)\, p_2 = 0 \end{array} \right.$$

The former system has a solution $(p_1, p_2) \neq (0, 0)$ if

$$\det \begin{pmatrix} a_{11-}\lambda & a_{12} \\ a_{12} & a_{22} - \lambda \end{pmatrix} = 0 \Longleftrightarrow \lambda^2 - (a_{11} + a_{22})\lambda + \det A_{00} = 0.$$

Notice that it is the characteristic equation of the matrix A_{00} which is diagonalizable.

Conclusion: If (v_1, v_2) is an eigenvector associated to the eigenvalue λ_1 of A_{00} then $Q[0, v_1, v_2]$ and $P[0, -v_2, v_1]$ satisfy the system

$$\begin{cases} p_1 q_1 + p_2 q_2 = 0 \\ P^t A Q = 0 \end{cases}$$

so its polar lines are the axes of the conic $\bar{C}$.

Last, we call *vertex* of a conic $\bar{C}$ to an intersection point of the axes of the conic with the conic.

Example 1

Let us consider the conic $\bar{C} \equiv x_0^2 + x_1^2 + x_2^2 - 2x_0x_2 + 2x_1x_2 = 0$. Answer the following questions:

1. Calculate the axes of the conic and represent them in the affine plane along with the affine conic.
2. Find the center of the conic.
3. Calculate the vertex of the conic.

Classificaction:

The matrix of the conic is

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

The determinant of A is: $\det(A) = -1 \neq 0$, ($\bar{C}$ is a regular conic) and as $\det A_{00} = 0$, the affine conic C is a parabola.

The eigenvalues of A_{00} are $\lambda_1 = 0$, $\lambda_2 = 2$. And the eigenvectors associated to these eigenvalues are:

$$\begin{aligned} \overline{v}_1 &= (-1, 1) \text{ eigenvector associated to } \lambda = 0 \\ \overline{v}_2 &= (1, 1) \text{ eigenvector associated to } \lambda = 2 \end{aligned}$$

Therefore the axes of the conic are the polar lines of the points at infinity $P_1[0, -1, 1]$ and $P_2[0, 1, 1]$;

this is,

$$r_{P_1} \equiv (0,-1,1)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

$$r_{P_2} \equiv (0,1,1)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

Thus the axes of the conic are

$$r_{P_1} \equiv x_0 = 0,$$
$$r_{P_2} \equiv -x_0 + 2x_1 + 2x_2 = 0.$$

The diameters of the conic intersect in the center; in particular, the center is the intersection point of the axes of the conic:

$$\begin{cases} x_0 = 0 \\ -x_0 + 2x_1 + 2x_2 = 0 \end{cases} \Longleftrightarrow \begin{cases} x_0 = 0 \\ x_1 + x_2 = 0 \end{cases} \Longleftrightarrow Z[0,-1,1]$$

The parabola has an improper center.

The vertex of the conic are the intersection points of the conic with its axes. As $\bar{C}$ is a parabola, it has a point at infinity, which is precisely the center Z and it is also a vertex of the parabola (the vertex at infinity):

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 - 2x_0x_2 + 2x_1x_2 = 0 \\ x_0 = 0 \end{cases} \Longleftrightarrow \begin{cases} (x_1 + x_2)^2 = 0 \\ x_0 = 0 \end{cases} \Longleftrightarrow Z[0, -1, 1]$$

The other vertex is the intersection of the parabola with its proper axis:

$$\begin{cases} 1 + x_1^2 + x_2^2 - 2x_2 + 2x_1x_2 = 0 \\ -1 + 2x_1 + 2x_2 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 1 + (x_1 + x_2)^2 - 2x_2 = 0 \\ 1 = 2x_1 + 2x_2 \end{cases} \Longleftrightarrow \begin{cases} 1 + \frac{1}{4} - 2x_2 = 0 \\ 1 = 2x_1 + 2x_2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_2 = \frac{5}{8} \\ x_1 = \frac{1}{2} - x_2 = \frac{1}{2} - \frac{5}{8} = -\frac{1}{8} \end{cases} \Longleftrightarrow V\left[1, -\frac{1}{8}, \frac{5}{8}\right].$$

Example 2

Let us consider the conic $\bar{C} \equiv x_0^2 - 4x_1^2 + x_2^2 - 2x_0x_1 - 2x_0x_2 = 0$. Answer the following questions:

1. Classify the conic.
2. Calculate the asymptotes of the conic.
3. Calculate the axes of the conic.
4. Find the center of the conic.
5. Calculate the vertex of the conic.

Classification:

The matrix of the conic is

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The determinant of A is: $\det(A) = -1 \neq 0$, ($\bar{C}$ is a regular conic) and as $\det A_{00} = -4 < 0$, the conic is a hyperbola. The points at infinity of the hyperbola satisfy the following equations:

$$\begin{cases} x_0^2 - 4x_1^2 + x_2^2 - 2x_0x_1 - 2x_0x_2 = 0 \\ x_0 = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} -4x_1^2 + x_2^2 = 0 \\ x_0 = 0 \end{cases} \Longleftrightarrow \begin{cases} (x_2 + 2x_1)(x_2 - 2x_1) = 0 \\ x_0 = 0 \end{cases}$$

$$\Longleftrightarrow P_1[0,1,-2] \text{ y } P_2[0,1,2]$$

Therefore, the asymptotes of the conic are:

$$\begin{aligned} r_{P_1} &\equiv P_1^t A X = x_0 - 4x_1 - 2x_2 = 0, \\ r_{P_2} &\equiv P_2^t A X = -3x_0 - 4x_1 + 2x_2 = 0. \end{aligned}$$

To calculate the axes first we calculate the eigenvectors of the matrix A_{00}. The eigenvalues of A_{00} are $\lambda_1 = -4$, $\lambda_2 = 1$. And the eigenvectors associated to these eigenvaules are:

$$\begin{aligned} \overline{v}_1 &= (1,0) \text{ eigenvector associated to } \lambda = -4 \\ \overline{v}_2 &= (0,1) \text{ eigenvector associated to } \lambda = 1 \end{aligned}$$

Therefore the axes of the conic are the polar lines of the points at infinity $Q_1[0,1,0]$ and $Q_2[0,0,1]$; this is,

$$\begin{aligned} r_{Q_1} &\equiv (0,1,0) \begin{pmatrix} 1 & -1 & -1 \\ -1 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = -x_0 - 4x_1 = 0 \\ r_{Q_2} &\equiv (0,0,1) \begin{pmatrix} 1 & -1 & -1 \\ -1 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = -x_0 + x_2 = 0 \end{aligned}$$

The center of the conic is the point of intersection of the axes. As C is a hyperbola, its center is a proper point and its coordinates verify the following equations

$$\begin{cases} 1+4x_1=0 \\ 1-x_2=0 \end{cases} \Longrightarrow Z\left[1,-\frac{1}{4},1\right]$$

The vertex of the conic are the intersection points of the conic with its axes. Therefore,

$$\begin{cases} x_0^2-4x_1^2+x_2^2-2x_0x_1-2x_0x_2=0 \\ -x_0-4x_1=0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 16x_1^2-4x_1^2+x_2^2+8x_1^2+8x_1x_2=0 \\ x_0=-4x_1 \end{cases} \Longleftrightarrow \begin{cases} 20x_1^2+8x_1x_2+x_2^2=0 \\ x_0=-4x_1 \end{cases}$$

$$\underset{t=x_2/x_1}{\Longleftrightarrow} \begin{cases} 20+8t+t^2=0 \\ x_0=-4x_1 \end{cases}$$

As $20+8t+t^2=0$ does not have real soltions, the axis $-x_0-4x_1=0$ of the conic intersects the conic in two points at infinity. Let us see the intersection of the conic with the other axis:

$$\begin{cases} x_0^2-4x_1^2+x_2^2-2x_0x_1-2x_0x_2=0 \\ -x_0+x_2=0 \end{cases} \iff \begin{cases} 4x_1^2+2x_2x_1=0 \\ -x_0+x_2=0 \end{cases}$$

$$\iff \begin{cases} 2\left(2x_1+x_2\right)x_1=0 \\ -x_0+x_2=0 \end{cases}$$

Thus $V_1[1,0,1]$ and $V_2[1,-1/2,1]$ are the two proper and real vertices of the hyperbola.

2.7 Metric invariants of a conic

Let $\mathcal{R} = \{\mathcal{O}, B = (\overline{e}_1, \overline{e}_2)\}$ and $\mathcal{R}' = \{\mathcal{O}', B' = (\overline{e}'_1, \overline{e}'_2)\}$ be two orthonormal coordinate systems of the affine plane $\mathbb{A}_2$. Let C be a conic of the euclidean affine plane with associated matrix A with respect to the coordinate system $\mathcal{R}$ and matrix A' with respect to the coordinate system $\mathcal{R}'$, this is,

$$C_{\mathcal{R}} \equiv (x_0, x_1, x_2) \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

$$C_{\mathcal{R}'} \equiv (x'_0, x'_1, x'_2) \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{01} & b_{11} & b_{12} \\ b_{02} & b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \end{pmatrix} = 0$$

then, it verifies

$$\det(A) = \det(A')$$

$$\left\{ \begin{array}{l} \det A_{00} = \det A'_{00} \\ a_{11} + a_{22} = b_{11} + b_{22} \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \text{The eigenvalues of } A_{00} \\ \text{and } A'_{00} \text{ are the same.} \end{array} \right.$$

where

$$A_{00} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \text{ and } A'_{00} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}.$$

2.8 Reduced form of a regular conic

Let C be a conic which, with respect to an orthonormal coordinate system $\mathcal{R} = \{\mathcal{O}, B = (\overline{e}_1, \overline{e}_2)\}$, has equation: $C_{\mathcal{R}} \equiv X^T A X = 0$.

2.8.1 Conics with proper center: hyperbola and ellipse

If $\det(A_{00}) \neq 0$ then there exists an orthonormal coordinate system $\mathcal{R}' = \{\mathcal{O}', B' = (\bar{e}'_1, \bar{e}'_2)\}$ such that the matrix expression of the conic in the new coordinate system is:

$$C_{\mathcal{R}'} \equiv (x'_0, x'_1, x'_2) \begin{pmatrix} d_{00} & 0 & 0 \\ 0 & d_{11} & 0 \\ 0 & 0 & d_{22} \end{pmatrix} \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \end{pmatrix} = 0$$

The equation $C_{\mathcal{R}'} \equiv d_{00}(x'_0)^2 + d_{11}(x'_1)^2 + d_{22}(x'_2)^2 = 0$ is called *reduced equation of the conic*, where

- $\mathcal{O}'$ is the *center* of the conic
- d_{11}, d_{22} are the eigenvalues of A_{00}
- $\vec{e}'_1, \vec{e}'_2$ are the eigenvectors of A_{00} (vectors with the direction of the axes of $\mathcal{C}$
- d_{00} verifies that $\det(A) = d_{00}d_{11}d_{22}$

Example 1

Let us consider the conic $\bar{C} \equiv 2x_0x_2 - 2x_1x_2 - x_0^2 + 7x_1^2 + 7x_2^2 = 0$.

Classification:

The matrix of the conic is:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 7 & -1 \\ 1 & -1 & 7 \end{pmatrix}.$$

The determinant of A is: $\det(A) = -55 \neq 0$, therefore, it is a regular conic. The eigenvalues of A_{00} are $\lambda_1 = 6$, $\lambda_2 = 8$ (thus, $\det A_{00} = \lambda_1\lambda_2 = 48 > 0$). The conic C is an ellipse.

Notable elements:

The *center* of the ellipse is a proper point which is not contained in the conic. We have:

$$P = A^{-1}U \text{ where } U[1,0,0]$$

so

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 7 & -1 \\ 1 & -1 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{48}{55} \\ \frac{1}{55} \\ \frac{7}{55} \end{pmatrix}.$$

This is

$$\text{Center} \equiv \left[\left(-\frac{48}{55}, \frac{1}{55}, \frac{7}{55}\right)\right] = \left[\left(1, -\frac{1}{48}, -\frac{7}{48}\right)\right]$$

The *diameters* of the ellipse are the lines that contain its center. The family of diameters is

$$\begin{vmatrix} x_0 & -\frac{48}{55} & 0 \\ x_1 & \frac{1}{55} & a \\ x_2 & \frac{7}{55} & b \end{vmatrix} \;=\; \frac{1}{55}\,(b-7a)\,x_0 + \frac{48}{55}bx_1 - \frac{48}{55}ax_2 = 0,$$

this is, $d_{a,b} \;\equiv\; (b-7a)\,x_0 + 48bx_1 - 48ax_2 = 0.$

Similarly they are the polar lines of points at infinity. If $P_\infty = [(0,\alpha,\beta)] \in r_\infty$, then its polar line has the following equation

$$r_{P_\infty} \;\equiv\; (0,\alpha,\beta)\begin{pmatrix} -1 & 0 & 1 \\ 0 & 7 & -1 \\ 1 & -1 & 7 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

this is, $d_{P_\infty} \;\equiv\; \beta x_0 + (7\alpha-\beta)\,x_1 + (7\beta-\alpha)\,x_2.$

Notice that if we take $\alpha = 7b - a$ and $\beta = b - 7a$, we have: $d_{P_\infty} \equiv d_{a,b}$.
The ellipse has no *asymptotes* because all its points are proper.

The *axes* of the ellipse contain the center and have directions given by two orthogonal eigenvectors. The eigenvectors of C are:

$$\begin{aligned}\bar{e}_1 &= (1,1) \text{ eigenvector associated to } \lambda_1 = 6,\\ \bar{e}_2 &= (-1,1) \text{ eigenvector associated to } \lambda_2 = 8,\end{aligned}$$

therefore, the axes are:

$$\text{Axis 1} \equiv \begin{vmatrix} x_0 & 1 & 0 \\ x_1 & \frac{1}{-48} & 1 \\ x_2 & \frac{7}{-48} & 1 \end{vmatrix} = \frac{1}{8}x_0 - x_1 + x_2 = 0,$$

$$\text{Axis 2} \equiv \begin{vmatrix} x_0 & 1 & 0 \\ x_1 & \frac{1}{-48} & -1 \\ x_2 & \frac{7}{-48} & 1 \end{vmatrix} = -\frac{1}{6}x_0 - x_1 - x_2 = 0.$$

The *vertices* are the points of intersection of the ellipse with its axes. As all the points of the ellipse are proper, we have to find the vertex in $x_0 = 1$, this is, we consider the systems

$$\bar{C} \cap \text{Axis } 1 \ \equiv \ \begin{cases} 2x_0x_2 - 2x_1x_2 - x_0^2 + 7x_1^2 + 7x_2^2 = 0 \\ \frac{1}{8}x_0 - x_1 + x_2 = 0 \end{cases}$$

$$\bar{C} \cap \text{Axis } 2 \ \equiv \ \begin{cases} 2x_0x_2 - 2x_1x_2 - x_0^2 + 7x_1^2 + 7x_2^2 = 0 \\ -\frac{1}{6}x_0 - x_1 - x_2 = 0 \end{cases}$$

for $x_0 = 1$ and we obtain

$$\begin{cases} 2x_2 - 2x_1x_2 - 1 + 7x_1^2 + 7x_2^2 = 0 \\ \frac{1}{8} - x_1 + x_2 = 0 \end{cases}$$

$$\Longrightarrow \ V_1^{\pm} = \left[\left(1, -\frac{1}{48} \pm \frac{1}{24}\sqrt{55}, -\frac{7}{48} \pm \frac{1}{24}\sqrt{55}\right)\right]$$

$$\begin{cases} 2x_2 - 2x_1x_2 - 1 + 7x_1^2 + 7x_2^2 = 0 \\ -\frac{1}{6} - x_1 - x_2 = 0 \end{cases}$$

$$\Longrightarrow \ V_2^{\pm} = \left[\left(1, -\frac{1}{48} \pm \frac{1}{48}\sqrt{165}, -\frac{7}{48} \mp \frac{1}{48}\sqrt{165}\right)\right].$$

Reduced form: the reduced equation of the ellipse is

$$C_{\mathcal{R}'} \equiv d_{00}(x_0')^2 + d_{11}(x_1')^2 + d_{22}(x_2')^2 = 0,$$

where $d_{11} = 6$, $d_{22} = 8$ and as $\det(A) = -55 = d_{00}d_{11}d_{22} = d_{00}48$, then $d_{00} = -55/48$. Therefore,

$$C_{\mathcal{R}'} \equiv \frac{-55}{48}(x_0')^2 + 6(x_1')^2 + 8(x_2')^2 = 0,$$

where the origin of the coordinate system $\mathcal{R}'$, is the center of the conic: $\mathcal{O}' = \left(-\frac{1}{48}, -\frac{7}{48}\right)$ and the basis is

$$B' = \left(\frac{\overline{e}_1}{\|\overline{e}_1\|}, \frac{\overline{e}_2}{\|\overline{e}_2\|}\right) = \left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right).$$

Using the matrix of the change of coordinate system, we have:

$$\begin{pmatrix} 1 & -\frac{1}{48} & -\frac{7}{48} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{48} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{7}{48} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{55}{48} & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Example 2

Let us consider the conic $\bar{C} \equiv 11x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + 8\sqrt{2}x_0x_2 + 3x_1x_2 = 0$.

Classification:

The matrix of the conic is

$$A = \begin{pmatrix} 11 & 0 & 4\sqrt{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 4\sqrt{2} & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

The determinant of A is: $\det(A) = -6 \neq 0$ (it is a regular conic) and the eigenvalues of A_{00} are $\lambda_1 = 1$, $\lambda_2 = -2$; therefore, $\det A_{00} = -2 < 0$. The conic C is a hyperbola.

Notable elements:

The *center* of the hyperbola is a proper point that it is not contained in the conic. We have:

$$P = A^{-1}U \text{ where } U[(1,0,0)]$$

then

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 11 & 0 & 4\sqrt{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 4\sqrt{2} & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\sqrt{2} \\ -\frac{1}{3}\sqrt{2} \end{pmatrix}.$$

The center is

$$Z \equiv \left[\left(\frac{1}{3}, -\sqrt{2}, -\frac{1}{3}\sqrt{2}\right)\right] = \left[\left(1, -3\sqrt{2}, -\sqrt{2}\right)\right].$$

The *diameters* of the hyperbola are the lines that contain its center (the polar lines of the points at infinity). The family of diameters is

$$\begin{vmatrix} x_0 & 1 & 0 \\ x_1 & -3\sqrt{2} & a \\ x_2 & -\sqrt{2} & b \end{vmatrix} = \sqrt{2}\,(a-3b)\,x_0 - bx_1 + ax_2 = 0,$$

this is, $d_{a,b} \equiv \sqrt{2}\,(a-3b)\,x_0 - bx_1 + ax_2 = 0.$

The hyperbola has two *asymptotes* which are tangent to the hyperbola in its infinity points. The infinity points of the hyperbola are:

$$P \in \bar{C} \cap r_\infty \Longrightarrow \begin{cases} -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + 3x_1x_2 = 0 \\ x_0 = 0 \end{cases} \Longrightarrow \begin{cases} P_1\left[\left(0,1,3-2\sqrt{2}\right)\right] \\ P_2\left[\left(0,1,3+2\sqrt{2}\right)\right] \end{cases}$$

The polar line of P_1 is:

$$r_1 \equiv \left(0,1,3-2\sqrt{2}\right)\begin{pmatrix} 11 & 0 & 4\sqrt{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 4\sqrt{2} & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Longrightarrow r_1 \equiv \left(12\sqrt{2}-16\right)x_0 + \left(4-3\sqrt{2}\right)x_1 + \sqrt{2}x_2 = 0,$$

and the polar line of P_2 is:

$$r_2 \equiv \left(0,1,3+2\sqrt{2}\right)\begin{pmatrix} 11 & 0 & 4\sqrt{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 4\sqrt{2} & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Longrightarrow r_2 \equiv \left(16+12\sqrt{2}\right)x_0 + \left(4+3\sqrt{2}\right)x_1 - \sqrt{2}x_2 = 0.$$

Thus the asymptotes of the hyperbola are

$$r_1 \equiv \left(12\sqrt{2}-16\right)x_0 + \left(4-3\sqrt{2}\right)x_1 + \sqrt{2}x_2 = 0,$$

$$r_2 \equiv \left(16+12\sqrt{2}\right)x_0 + \left(4+3\sqrt{2}\right)x_1 - \sqrt{2}x_2 = 0.$$

Notice that, for $a = 1$, $b = 3 - 2\sqrt{2}$ we have

$$\begin{aligned} d_{1,3-2\sqrt{2}} &\equiv \sqrt{2}\left(1 - 3(3 - 2\sqrt{2})\right)x_0 - (3 - 2\sqrt{2})x_1 + x_2 = 0, \\ &\equiv \left(12 - 8\sqrt{2}\right)x_0 - (3 - 2\sqrt{2})x_1 + x_2 = 0 \\ &\equiv \sqrt{2}\left(\left(12 - 8\sqrt{2}\right)x_0 - (3 - 2\sqrt{2})x_1 + x_2\right) = 0 \\ &\equiv \left(\left(12\sqrt{2} - 16\right)x_0 + (4 - 3\sqrt{2})x_1 + \sqrt{2}x_2\right) = 0 \\ &\equiv r_1 \end{aligned}$$

and for $a = 1$ and $b = 3 + 2\sqrt{2}$ we have: $d_{1,3+2\sqrt{2}} \equiv r_1$.

The *axes* of the hyperola contain the center and have directions given by two orthonormal eigenvectors. The eigenvectors of C are:

$$\begin{aligned} \bar{e}_1 &= (1,1) \text{ eigenvector associated to } \lambda_1 = 1, \\ \bar{e}_2 &= (-1,1) \text{ eigenvector associated to } \lambda_2 = -2, \end{aligned}$$

thus the axes are:

$$\begin{aligned} \text{Axis1} &\equiv \begin{vmatrix} x_0 & 1 & 0 \\ x_1 & -3\sqrt{2} & 1 \\ x_2 & -\sqrt{2} & 1 \end{vmatrix} = x_2 - x_1 - 2x_0\sqrt{2} = 0, \\ \text{Axis 2} &\equiv \begin{vmatrix} x_0 & 1 & 0 \\ x_1 & -3\sqrt{2} & -1 \\ x_2 & -\sqrt{2} & 1 \end{vmatrix} = -x_1 - x_2 - 4x_0\sqrt{2} = 0. \end{aligned}$$

The *vertices* are the points of intersection of the hyperbola with its axes

$$\bar{C} \cap \text{Axis } 1 \equiv \begin{cases} 11x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + 8x_0x_2 + 3x_1x_2 = 0 \\ x_2 - x_1 - 2x_0\sqrt{2} = 0 \end{cases}$$

$$\bar{C} \cap \text{Axis } 2 \equiv \begin{cases} 11x_0^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + 8x_0x_2 + 3x_1x_2 = 0 \\ -x_1 - x_2 - 4x_0\sqrt{2} = 0 \end{cases}$$

If $x_0 = 1$, then

$$\begin{cases} 11 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + 8x_2 + 3x_1x_2 = 0 \\ x_2 - x_1 - 2\sqrt{2} = 0 \end{cases}$$

there is not real solution

$$\begin{cases} 11 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + 8x_2 + 3x_1x_2 = 0 \\ -x_1 - x_2 - 4\sqrt{2} = 0 \end{cases}$$

$$\Longrightarrow V_2 = \left[\left(1, \; -1 - 2\sqrt{2} \pm \frac{1}{2}\sqrt{31 - 16\sqrt{2}}, \; 1 - 2\sqrt{2} \pm \frac{1}{2}\sqrt{31 - 16\sqrt{2}}\right)\right]$$

Reduced form: the reduced equation of this hyperbola is

$$d_{00}(x_0')^2 + d_{11}(x_1')^2 + d_{22}(x_2')^2 = 0$$

where $d_{11} = 1$, $d_{22} = -2$ and as $\det(A) = -6 = d_{00}d_{11}d_{22} = -2d_{00}$, then $d_{00} = 3$. Therefore,

$$C_{\mathcal{R}'} \equiv 3(x_0')^2 + (x_1')^2 - 2(x_2')^2 = 0,$$

where the origin of the coordinate system $\mathcal{R}'$, is the center of the conic: $\mathcal{O}' = \left(-3\sqrt{2}, -\sqrt{2}\right)$ and the basis is

$$B' = \left\{ \frac{\overline{e}_1}{\|\overline{e}_1\|}, \frac{\overline{e}_2}{\|\overline{e}_2\|} \right\} = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}.$$

Using the matrix of the change of coordinate system we have:

$$\begin{pmatrix} 1 & -3\sqrt{2} & -\sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ -3\sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

2.8.2 Conics with center at infinity: parabola

If $\det(A_{00}) = 0$ then there exists an orthonormal coordinate system $\mathcal{R}' = \{\mathcal{O}', B' = (\overline{e}'_1, \overline{e}'_2)\}$ such that the matrix expression of the conic in $\mathcal{R}'$ is:

$$C_{\mathcal{R}'} \equiv (x'_0, x'_1, x'_2) \begin{pmatrix} 0 & 0 & d_{02} \\ 0 & d_{11} & 0 \\ d_{02} & 0 & 0 \end{pmatrix} \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \end{pmatrix} = 0.$$

The equation $C_{\mathcal{R}'} \equiv d_{11}(x'_1)^2 + 2d_{02}x'_0x'_2 = 0$ is called *reduced equation of the conic*, where

$$\begin{cases} \mathcal{O}' \text{ is the } \textit{vertex} \text{ of the parabola} \\ d_{11}, 0 \text{ are the eigenvalues of } A_{00} \\ \overline{e}'_1, \overline{e}'_2 \text{ are eigenvectors associated to } d_{11}, 0 \text{ resp.} \end{cases}$$

If we change the order of the vectors, the matrix that we obtain is

$$\begin{pmatrix} 0 & d_{01} & 0 \\ d_{01} & 0 & 0 \\ 0 & 0 & d_{22} \end{pmatrix}.$$

Example

Let us consider the conic $\bar{C} \equiv -2x_0x_2 + 4x_1x_2 + x_0^2 + 4x_1^2 + x_2^2 = 0$.

Classification:

The matrix of the conic is

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

The determinant of A is: $\det(A) = -4 \neq 0$ (it is a regular conic) and the eigenvalues of A_{00} are $\lambda_1 = 0$, $\lambda_2 = 5$ (therefore, $\det A_{00} = \lambda_1\lambda_2 = 0$). The conic $\bar{C}$ is a parabola.

Notable elements:

The *center* of the parabola (the pole of the infinity line) is an improper point that is contained in the conic. We have:

$$\bar{C} \cap r_\infty \equiv 4x_1x_2 + 4x_1^2 + x_2^2 = 0 \underset{t=x_2/x_1}{\Longrightarrow} 4t + 4 + t^2 = 0 \Longrightarrow t = -2 \Longrightarrow Z[(0, 1, -2)]$$

The *diameters* of the parabola are all the lines that contain its center (polar lines of points at infinity). They have the direction of the vector $(0,1,-2)$, therefore,

$$\text{Family of diameters } d_a \equiv ax_0 + 2x_1 + x_2 = 0.$$

The *asymptote* of the parabola (tangent line in its point at infinity) is the infinity line: $x_0 = 0$.
The parabola has an unique *proper axis*. We have:

$$\begin{aligned} \overline{e}_1 &= (2,1) \text{ eigenvector associated to } \lambda_2 = 5, \\ \overline{e}_2 &= (-1/2,1) \text{ eigenvector associated to } \lambda_1 = 0. \end{aligned}$$

Therefore, the proper axis of the parabola is the polar line of the point: $P[(0,2,1)]$; this is

$$\text{Axis} \equiv (0,2,1) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \Longrightarrow \text{Axis} \equiv -x_0 + 10x_1 + 5x_2 = 0.$$

The *vertex* is the intersection of the parabola with its axis:

$$\bar{C} \cap \text{axis} \equiv \begin{cases} -2x_0x_2 + 4x_1x_2 + x_0^2 + 4x_1^2 + x_2^2 = 0 \\ -x_0 + 10x_1 + 5x_2 = 0 \end{cases}$$

In $x_0 = 1$

$$\begin{cases} -2x_2 + 4x_1x_2 + 1 + 4x_1^2 + x_2^2 = 0 \\ -1 + 10x_1 + 5x_2 = 0 \end{cases} \Longrightarrow V\left[\left(1, -\frac{4}{25}, \frac{13}{25}\right)\right]$$

Reduced form: The reduced equation of this parabola is

$$C_{\mathcal{R}'} \equiv d_{11}(x_1')^2 + 2d_{02}x_0'x_2' = 0$$

where $d_{11} = 5$ and as $\det(A) = -4 = (d_{02})^2 d_{11}$ then $(d_{02})^2 = 4/5$. Therefore,

$$C_{\mathcal{R}'} \equiv 5(x_1')^2 + \frac{4}{\sqrt{5}}x_0'x_2' = 0,$$

where the origin of the coordinate system $\mathcal{R}'$ is the vertex of the parabola: $\mathcal{O}' = \left(-\frac{4}{25}, \frac{13}{25}\right)$, and the basis is

$$B' = \left\{\frac{\overline{e}_1}{\|\overline{e}_1\|}, \frac{\overline{e}_2}{\|\overline{e}_2\|}\right\} = \left\{\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right\}.$$

Using the matrix of change of coordinate system, we have:

$$\begin{pmatrix} 1 & -\frac{4}{25} & \frac{13}{25} \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{25} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{13}{25} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{2}{5}\sqrt{5} \\ 0 & 5 & 0 \\ -\frac{2}{5}\sqrt{5} & 0 & 0 \end{pmatrix}.$$