CHAPTER II: AFFINE AND EUCLIDEAN GEOMETRY

Euclidean space

A real vector space E is an euclidean vector space if it is provided with an scalar product (or dot product); this is, a bilineal, symmetric and positive-definite map

$$\langle \ , \ \rangle : E \times E \longrightarrow \mathbb{R}.$$

We denote a scalar product by $\langle \overline{u}, \overline{v} \rangle$ or $\overline{u} \cdot \overline{v}$ indistinctly.

A scalar product defined in a vector space E allows the definition of a norm as follows:

$$\| \quad \| : E \longrightarrow \mathbb{R}, \ \| v \| = \sqrt{\langle v, v \rangle}$$

The *angle* between two non zero vectors $\overline{u}$ and $\overline{v}$ of an euclidean vector space E, is the real number in $[0, \pi]$ that we will denote by $\widehat{(\overline{u}, \ \overline{v})}$ and such that

$$\cos\widehat{(\overline{u}, \ \overline{v})} = \frac{\overline{u}_1 \cdot \overline{u}_2}{\|\overline{u}_1\| \, \|\overline{u}_2\|}.$$

3. AFFINE EUCLIDEAN SPACE

Definition An affine space $(\mathbb{A}, V, \phi)$ is an euclidean affine space if the vector space V is an euclidean vector space.

Notation We will denote an euclidean vector space by E and an euclidean affine space by $(\mathbb{E}, E, \phi)$.

Definition A distance d on an affine space $\mathbb{A}$ is a map

$$d\colon \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{R},\ (P, Q) \longmapsto d(P, Q)$$

that verifies:

1. d is positive-definite; this is, $d(P, Q) \geq 0$ and $d(P, Q) = 0$ if and only if $P = Q$.
2. d is symmetric; this is, $d(P, Q) = d(Q, P)$.
3. d verifies the triangle inequality; this is,$d(P, Q) \leq d(P, R) + d(R, Q)$.

3.1 Orthogonal coordinate systems

An affine coordinate system $\mathcal{R} = \{O; \{\overline{e}_1, \ldots, \overline{e}_n\}\}$ in an euclidean affine space $(\mathbb{E}, E, \phi)$ is called orthogonal (resp. orthonormal), if the basis $B = \{\overline{e}_1, \ldots, \overline{e}_n\}$ of the vector space V is orthogonal (resp. orthonormal).

Change of orthonormal coordinate system

Let $(\mathbb{E}, E, \phi)$ be an euclidean affine space of dimension n. Let $\mathcal{R} = \{O; B\}$ and $\mathcal{R}' = \{O'; B'\}$ be two orthonormal coordinate systems of $\mathbb{E}$.

If $O'(a_1, \ldots, a_n)$ and $M(B', B)$ is the matrix of change of basis then the matrix of the change of coordinate system from $\mathcal{R}'$ to $\mathcal{R}$ is:

$$M_f(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & & & \\ \vdots & & M(B', B) & \\ a_n & & & \end{pmatrix}$$

The following statements hold:

1. The matrix $M(B', B)$ is an orthogonal matrix; this is, $M(B', B)^{-1} = M(B', B)^t$.
2. $\det(M(B', B) = \pm 1$. If $\det(M(B', B) = 1$ we say that B' and B have the same orientation and if $\det(M(B', B) = -1$ we say that B' and B have different orientation.

3.2 Orthogonal affine subspaces

Let $(\mathbb{E}, E, \phi)$ be an euclidean affine space of dimension n.
We must remember that, given a vector subspace $W \subset E$, the set defined as follows:

$$\{\overline{v} \in E \mid \overline{v} \cdot \overline{w} = 0 \text{ for every } \overline{w} \in W\}$$

is a vector subspace of E that we denote $W^{\perp}$ and call orthogonal subpace to W. It holds

$$E = W \oplus W^{\perp}.$$

Therefore,

$$\dim E = \dim W + \dim W^{\perp}.$$

Definition

- Two affine subspaces L_1 and L_2 of $\mathbb{E}$ such that $\dim \overline{L}_1 + \dim \overline{L}_2 < n$ are *orthogonal* if their respective associated vector subspaces $\overline{L}_1$ and $\overline{L}_2$ are orthogonal; this is, any vector $\overline{u} \in \overline{L}_1$ is orthogonal to any vector $\overline{v} \in \overline{L}_2$.

  If $L_1 = P_1 + \langle \overline{u}_1, \ldots, \overline{u}_s \rangle$ and $L_2 = P_2 + \langle \overline{v}_1, \ldots, \overline{v}_r \rangle$ then L_1 and L_2 are orthogonal if $\overline{u}_i \cdot \overline{v}_j = 0$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r$.

- If $\dim \overline{L}_1 + \dim \overline{L}_2 \geq n$, we will say that L_1, L_2 are *orthogonal* if $\overline{L_1}^{\perp}$ and $\overline{L_2}^{\perp}$ are orthogonal.

Notation. If L_1 and L_2 are orthogonal, we will write $L_1 \perp L_2$.

Definition An affine subspace L' with associated vector subspace $\overline{L}'$ is called *orthogonal* to an affine subspace L with associated vector subspace $\overline{L}$ if $\overline{L}$ and $\overline{L'}$ are orthogonal and in addition $V = \overline{L} \oplus \overline{L}'$.

Particular cases

1. Two lines $r = P + \langle \overline{v} \rangle$, $r' = P' + \langle \overline{v}' \rangle$ are orthogonal if and only if $\overline{v} \cdot \overline{v}' = 0$.
2. In dimension 3, a line $r = P + \langle \overline{v} \rangle$ is the orthogonal subspace to a plane with associated vector subspace W if $\overline{v}$ is orthogonal to any vector of W (in this case, $V = W \oplus \langle \overline{v} \rangle$).
3. Let $\pi = P + \langle \overline{u}_1, \overline{u}_2 \rangle$ be an affine plane. The line $r = P + \langle \overline{v} \rangle$ is orthogonal to π if the vector $\overline{v}$ is orthogonal to vectors $\overline{u}_1$ and $\overline{u}_2$.
4. In dimension 3, a line $r = P + \langle \overline{v} \rangle$ is orthogonal to a plane $\pi = P + \langle \overline{u}_1, \overline{u}_2 \rangle$ if the vector $\overline{v}$ is parallel to the normal vector to the plane; this is, $\overline{v}$ and $\overline{n}$ are parallel, where $\overline{n} = \overline{u}_1 \wedge \overline{u}_2$ and $\wedge$ denotes the cross product in $\mathbb{E}_3$.
5. In dimension 3, two planes π_1 and π_2 are orthogonal if their respective normal vectors are orthogonal.

3.2.1 Orthogonal projection of a point on an affine subspace

Let L be an affine subspace of an euclidean affine space $\mathbb{E}$ and let P be a point of $\mathbb{E}$ that does not belong to L (this is, $P \in \mathbb{E} \diagdown L$).

The *orthogonal projection of* P *on* L is P_0, the point of intersection with L of the orthogonal subspace to L containing P.

3.3 Distance between two affine subspaces

Let $(\mathbb{E}, E, \phi)$ be an euclidean affine subspace of dimension n. Let L_1 and L_2 be two affine subspaces of $\mathbb{E}$. We define the *distance between* L_1 and L_2 as the minimum of the distances between its points; this is,

$$d(L_1, L_2) = \min \{d(P_1, P_2) \mid P_1 \in L_1 \text{ and } P_2 \in L_2\}.$$

Note that if $L_1 \cap L_2 \neq \emptyset$ then $d(L_1, L_2) = 0$.

- If L_1 and L_2 are parallel subspaces, let us suppose that $\overline{L}_1 \subset \overline{L}_2$ then

$$d(L_1, L_2) = d(P, L_2) = \min \{d(P, P_2) \mid P_2 \in L_2\}$$

  where P is an arbitrary point of L_1.

- If $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ are not parallel then we build a subspace H, which is parallel to one of them and contains the other. For example, we can take $H = P_1 + \overline{L}_1 + \overline{L}_2$. The subspace H contains L_1 and it is parallel to L_2; therefore,

$$d(L_1, L_2) = d(H, L_2)$$

  and we are in the first case.

Thus, the problem is reduced to computing the distance from a point P to a subspace L.

3.3.1 Distance between a point P and an affine subspace L

Let $(\mathbb{E}, E, \phi)$ be an euclidean affine space of dimension n. Let $P \in \mathbb{E}$ and let $L = Q + \overline{L}$ be an affine subspace of $\mathbb{E}$, with $P \notin L$. Then, if we call P_0 the orthogonal projection of P on L, we have:

$$d(P, L) = d(P, P_0) = \left\| \overline{PP_0} \right\| .$$

Now we will study some particular cases of distance between affine subspaces.

Distance between a point P and a hyperplane H

Let P be a point with coordinates $(p_1, \ldots, p_n)$ and let H be the hyperplane with cartesian equation $a_1x_1 + \cdots + a_nx_n + b = 0$.

If we denote the orthogonal projection of P on H by P_0 we have:

$$d(P, H) = d(P, P_0).$$

Let $\overline{u}$ be the unit vector normal to the hyperplane; this is,

$$\overline{u} = \frac{(a_1, \ldots, a_n)}{\sqrt{a_1^2 + \cdots + a_n^2}}$$

The following formula hold:

$$\begin{aligned} d(P, P_0) &= |\overline{PP_0} \cdot \overline{u}| = \left| (x_1 - p_1, \ldots, x_n - p_n) \cdot \frac{(a_1, \ldots, a_n)}{\sqrt{a_1^2 + \cdots + a_n^2}} \right| \\ &= \frac{|a_1 x_1 + \cdots + a_1 x_n - (a_1 p_1 + \cdots + a_1 p_n)|}{\sqrt{a_1^2 + \cdots + a_n^2}} \\ &= \frac{|a_1 p_1 + \cdots + a_1 p_n + b|}{\sqrt{a_1^2 + \cdots + a_n^2}} \end{aligned}$$

Distance between a point P and a line r

Let us consider $P \in \mathbb{E}$ and let $r \equiv Q + \langle \overline{u} \rangle$ be a line in $\mathbb{E}$. By P_0 we denote the orthogonal projection of P on r, then we have:

$$d(P, r) = d(P, P_0),$$

where P_0 is a point of the line r and it holds $\overline{PP_0} \cdot \overline{u} = 0$.

Distance between two skew lines in $\mathbb{E}_3$

Let $r_1 \equiv P_1 + \langle \overline{u}_1 \rangle$ and $r_2 \equiv P_2 + \langle \overline{u}_2 \rangle$ be two lines in $\mathbb{E}_3$. Let us build a plane parallel to one of them, which contains the other one; for example, the plane,

$$\pi \equiv P_2 + \langle \overline{u}_1, \overline{u}_2 \rangle$$

is parallel to the line r_1 and contains the line r_2.

Also, let us consider the unit vector normal to the plane π; this is, the vector

$$\overline{u} = \frac{1}{\|\overline{u}_1 \wedge \overline{u}_2\|}\overline{u}_1 \wedge \overline{u}_2$$

where $\wedge$ denotes the cross product in $\mathbb{E}_3$. We have:

$$d(r_1, r_2) = d(r_1, \pi)$$

Let us consider the parallelepiped whose edges are vectors $\overline{P_2P_1}$, $\overline{u}_1$ and $\overline{u}_2$. The volume of the mentioned parallelepiped is the absolute value of the triple product of $\overline{u}_1$, $\overline{u}_2$ and $\overline{P_2P_1}$; this is,

$$V = \left|\left[\overline{u}_1, \overline{u}_2, \overline{P_2P_1}\right]\right| = \left|\overline{P_2P_1} \cdot (\overline{u}_1 \wedge \overline{u}_2)\right| = \left\|\overline{P_2P_1}\right\| \|\overline{u}_1 \wedge \overline{u}_2\| \, |\cos\alpha|$$

where α is the angle formed by vectors $\overline{P_2P_1}$ and $\overline{u}_1 \wedge \overline{u}_2$.

The area of the base of the parallelepiped is:

$$A = \|\overline{u}_1 \wedge \overline{u}_2\|$$

The distance between r_1 and π is the height of the above mentioned parallelepiped. Therefore

$$d(r_1, r_2) = d(r_1, \pi) = \frac{\left|\left[\overline{u}_1, \overline{u}_2, \overline{P_2P_1}\right]\right|}{\|\overline{u}_1 \wedge \overline{u}_2\|} = \left\|\overline{P_2P_1}\right\| |\cos\alpha| \, .$$