CLASSIFICATION OF QUADRICS

Quadrics with proper center. det(A00)≠0

In an appropriate coordinate system the matrix of the quadric is:

```
> restart:with(linalg):
> A:=matrix(4,4,[d0,0,0,0,0,lambda[1],0,0,0,0,lambda[2],0,0,0,0,
  lambda[3]]);
```

$$A := \begin{bmatrix} d0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \quad (1.1)$$

```
> A00:=submatrix(A,2..4,2..4);
```

$$A00 := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (1.2)$$

```
> det(A00);
```

$$\lambda_1 \lambda_2 \lambda_3 \quad (1.3)$$

```
> det(A);
```

$$d0\, \lambda_1 \lambda_2 \lambda_3 \quad (1.4)$$

```
> X:=matrix(1,4,[x0,x1,x2,x3]):
> red_equ:=simplify(evalm(X&*A&*transpose(X)))[1,1];
```

$$red\_equ := x0^2\, d0 + x1^2\, \lambda_1 + x2^2\, \lambda_2 + x3^2\, \lambda_3 \quad (1.5)$$

NON DEGENERATE, rank(A)=4

Ellipsoids. Sig(A00)=3

Example of an IMAGINARY ELLIPSOID. det(A)>0

```
> lambda[1]:=1/4;lambda[2]:=1/9;lambda[3]:=1/25;
```

$$\lambda_1 := \frac{1}{4}$$

$$\lambda_2 := \frac{1}{9}$$

$$\lambda_3 := \frac{1}{25} \tag{1.1.1}$$

```
> d0:=4;
```

$$d0 := 4 \tag{1.1.2}$$

```
> red_equ;
```

$$4\,x0^2 + \frac{1}{4}\,x1^2 + \frac{1}{9}\,x2^2 + \frac{1}{25}\,x3^2 \tag{1.1.3}$$

```
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := 4 + \frac{1}{4}\,x1^2 + \frac{1}{9}\,x2^2 + \frac{1}{25}\,x3^2 \tag{1.1.4}$$

It is an imaginary quadric, there are no real points.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-5..5,x2=-5..5,x3=-5..5);
```

Example of a REAL ELLIPSOID. det(A)<0

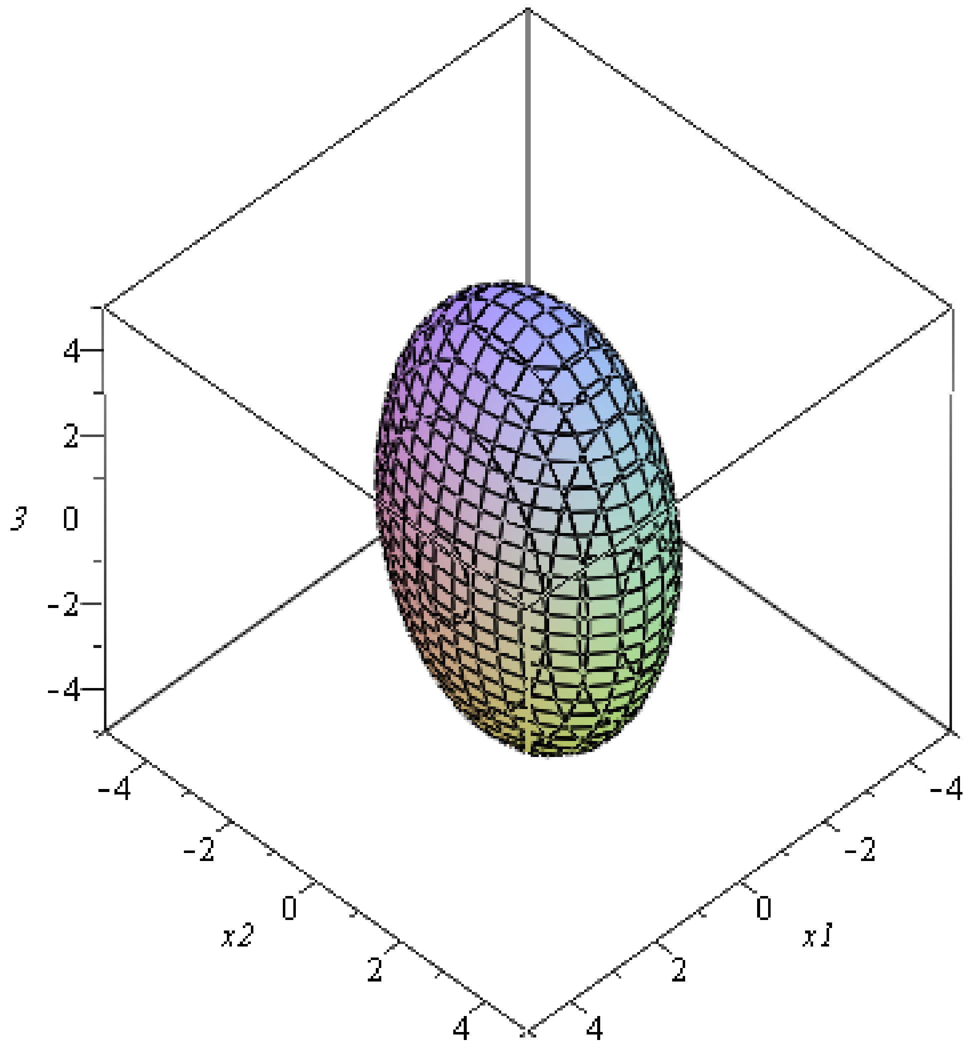
```
> d0:=-1;
```

$$d0 := -1 \tag{1.1.5}$$

```
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -1 + \frac{1}{4}\,x1^2 + \frac{1}{9}\,x2^2 + \frac{1}{25}\,x3^2 \qquad \textbf{(1.1.6)}$$

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-5..5,x2=-5..5,x3=-5..5,
  numpoints=10000,axes=boxed);
```



Hyperboloids. Sig(A00)=1

Example of a HYPERBOLIC HYPERBOLOID. det(A)>0

```
> lambda[1]:=-1/4;lambda[2]:=1/9;lambda[3]:=1/25;
```

$$\lambda_1 := -\frac{1}{4}$$

$$\lambda_2 := \frac{1}{9}$$

(1.2.1)

$$\lambda_3 := \frac{1}{25} \tag{1.2.1}$$

```
> d0:=-1;
```

$$d0 := -1 \tag{1.2.2}$$

```
> red_equ;
```

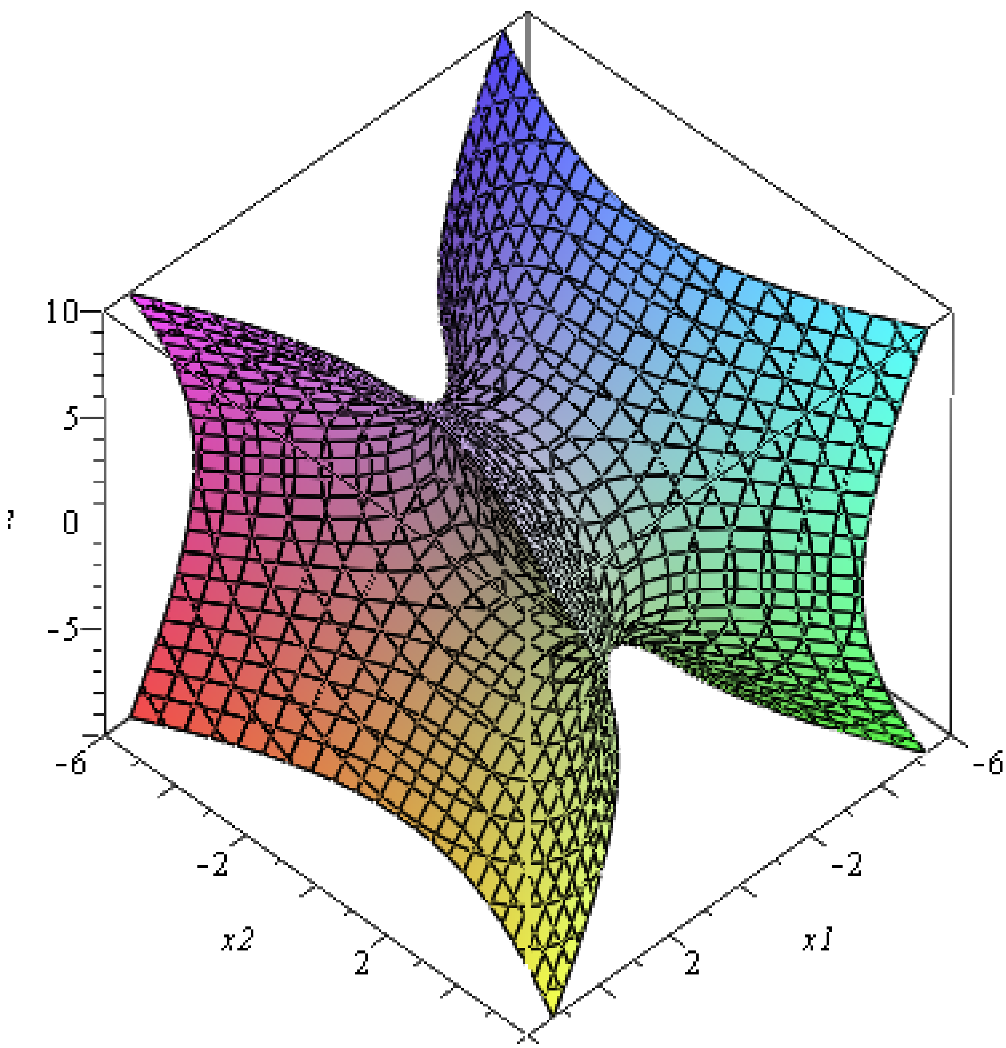
$$-x0^2 - \frac{1}{4}\,x1^2 + \frac{1}{9}\,x2^2 + \frac{1}{25}\,x3^2 \tag{1.2.3}$$

```
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -1 - \frac{1}{4}\,x1^2 + \frac{1}{9}\,x2^2 + \frac{1}{25}\,x3^2 \tag{1.2.4}$$

It contains lines, it is a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-6..6,x2=-6..6,x3=-10..10,
  numpoints=10000,axes=boxed);
```



Example of ELLIPTIC HYPERBOLOID. det(A)<0

```
> lambda[1]:=-1/4;lambda[2]:=1/9;lambda[3]:=1/25;
```

$$\lambda_1 := -\frac{1}{4}$$

$$\lambda_2 := \frac{1}{9}$$

$$\lambda_3 := \frac{1}{25} \tag{1.2.5}$$

```
> d0:=1;
```

$$d0 := 1 \tag{1.2.6}$$

```
> red_equ;
```

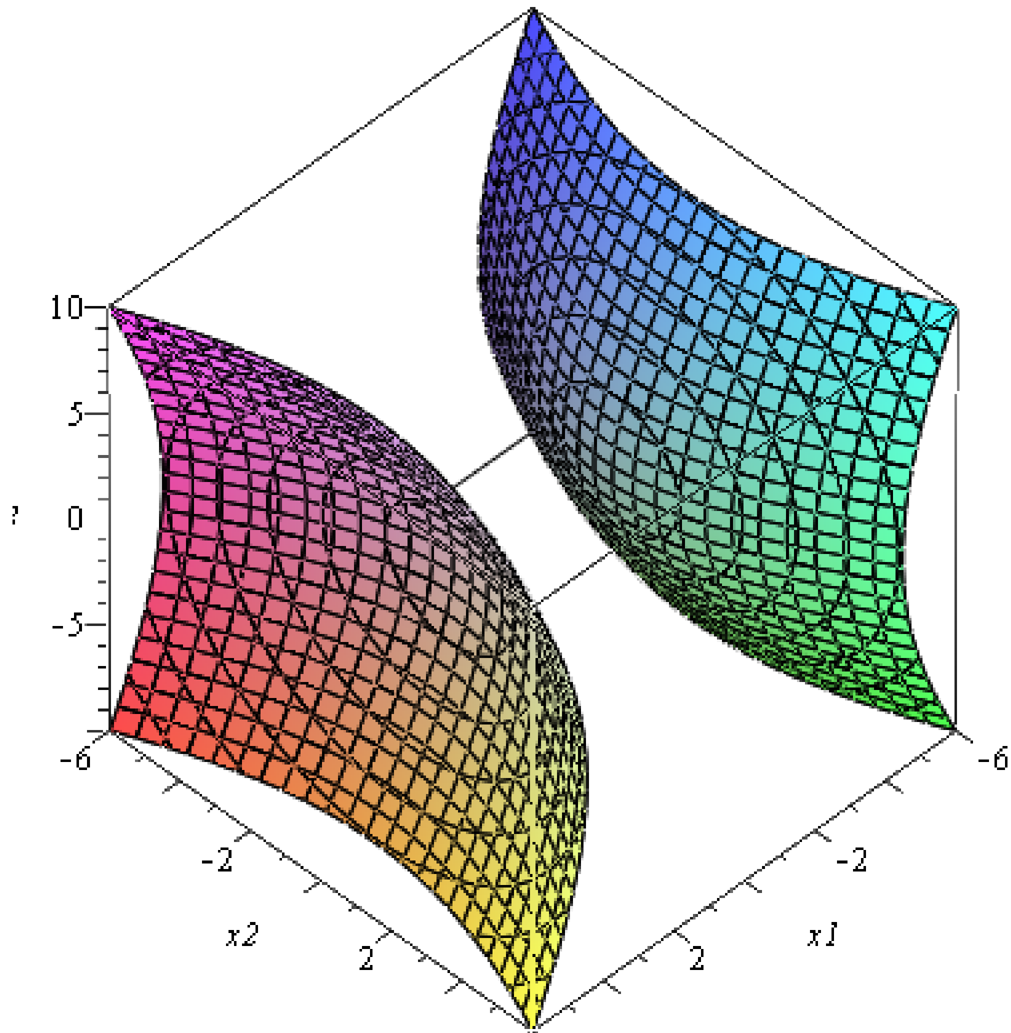
$$x0^2 - \frac{1}{4}\, x1^2 + \frac{1}{9}\, x2^2 + \frac{1}{25}\, x3^2 \tag{1.2.7}$$

```
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := 1 - \frac{1}{4}\, x1^2 + \frac{1}{9}\, x2^2 + \frac{1}{25}\, x3^2 \tag{1.2.8}$$

It does not contain lines, it is not a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-6..6,x2=-6..6,x3=-10..10,
  numpoints=10000,axes=boxed);
```



DEGENERATE, rank(A)=3

Cones

Example of IMAGINARY CONE with one real point (the singular point). Sig(A00)=3

```
> lambda[1]:=1/4;lambda[2]:=1/9;lambda[3]:=1/25;
```

$$\lambda_1 := \frac{1}{4}$$

$$\lambda_2 := \frac{1}{9}$$

$$\lambda_3 := \frac{1}{25} \quad \textbf{(1.3.1)}$$

```
> d0:=0;
```

$$d0 := 0 \quad \textbf{(1.3.2)}$$

```
> red_equ;
```

(1.3.3)

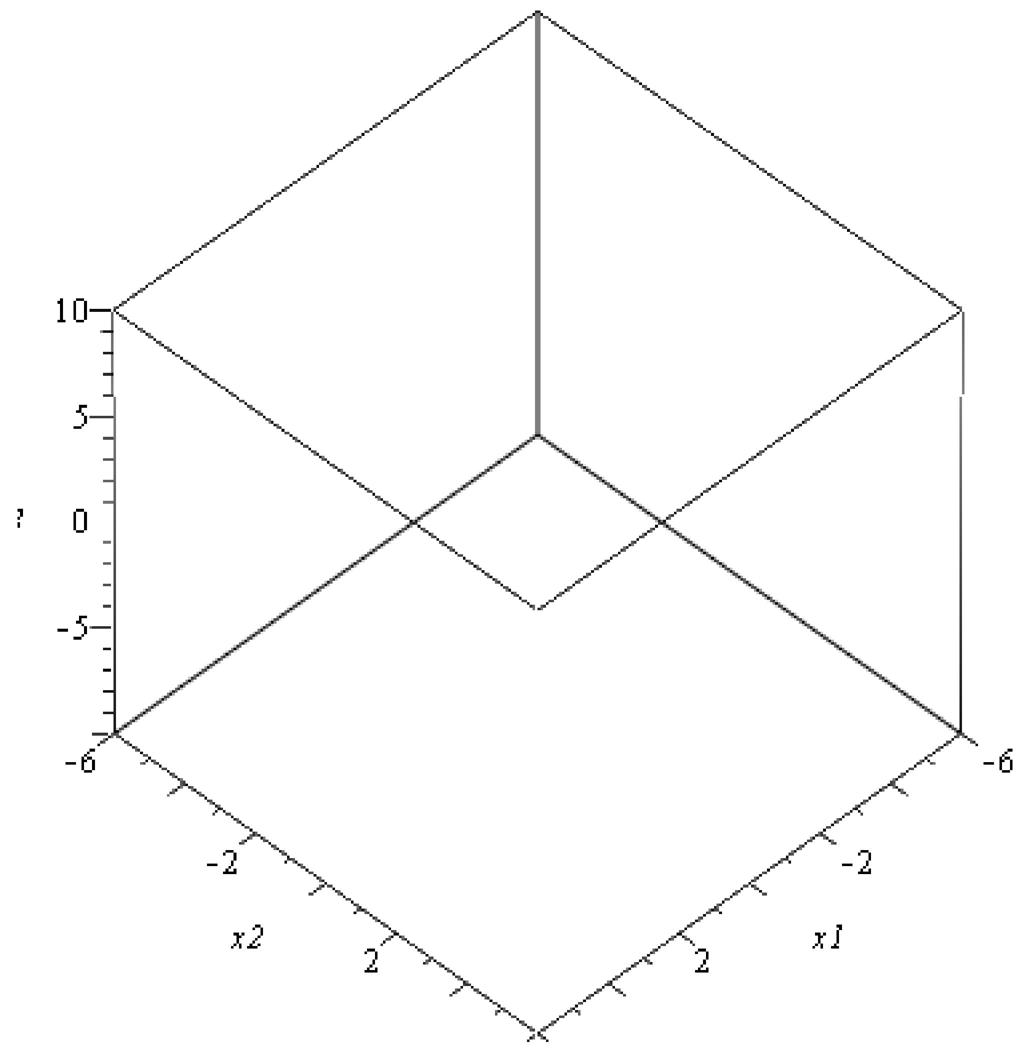
$$\frac{1}{4}\,x1^2+\frac{1}{9}\,x2^2+\frac{1}{25}\,x3^2 \qquad \textbf{(1.3.3)}$$

```
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := \frac{1}{4}\,x1^2+\frac{1}{9}\,x2^2+\frac{1}{25}\,x3^2 \qquad \textbf{(1.3.4)}$$

In a graph with implicitplot3d we would not see the real point.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-6..6,x2=-6..6,x3=-10..10,
  numpoints=10000,axes=boxed);
```



Example of a REAL CONE. Sig(A00)=1.

```
> lambda[1]:=-1/9;lambda[2]:=1/9;lambda[3]:=1/9;
```

$$\lambda_1 := -\frac{1}{9}$$

$$\lambda_2 := \frac{1}{9}$$

$$\lambda_3 := \frac{1}{9} \qquad \textbf{(1.3.5)}$$

```
> d0:=0;
```

$$d0 := 0 \qquad (1.3.6)$$

```
> red_equ;
```

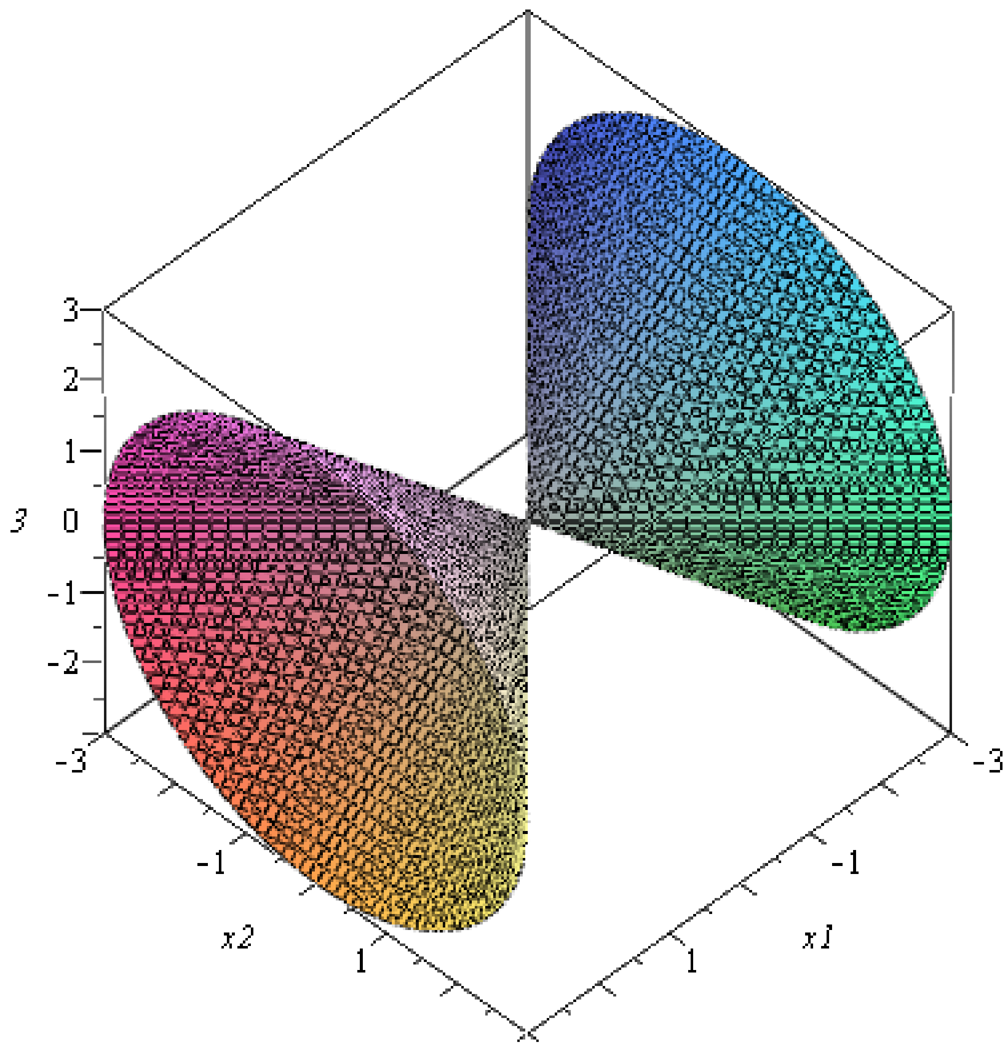
$$-\frac{1}{9}\,x1^2+\frac{1}{9}\,x2^2+\frac{1}{9}\,x3^2 \qquad (1.3.7)$$

```
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -\frac{1}{9}\,x1^2+\frac{1}{9}\,x2^2+\frac{1}{9}\,x3^2 \qquad (1.3.8)$$

It contains lines, it is a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-3..3,x2=-3..3,x3=-3..3,
  numpoints=100000,axes=boxed);
```



Quadrics with improper center. det(A00)=0

In an appropriate coordinate system the matrix of the quadric is:

```
> restart:with(linalg):
> A:=matrix(4,4,[0,0,0,b03,0,b11,0,0,0,0,b22,0,b03,0,0,0]);
```

$$A := \begin{bmatrix} 0 & 0 & 0 & b03 \\ 0 & b11 & 0 & 0 \\ 0 & 0 & b22 & 0 \\ b03 & 0 & 0 & 0 \end{bmatrix} \quad (2.1)$$

```
> A00:=submatrix(A,2..4,2..4);
```

$$A00 := \begin{bmatrix} b11 & 0 & 0 \\ 0 & b22 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.2)$$

```
> det(A00)=b11*b22*0:
> det(A);
```

$$-b03^2\, b11\, b22 \quad (2.3)$$

```
> X:=matrix(1,4,[x0,x1,x2,x3]):
> red_equ:=simplify(evalm(X&*A&*transpose(X)))[1,1];
```

$$red\_equ := 2\, x3\, b03\, x0 + x1^2\, b11 + x2^2\, b22 \quad (2.4)$$

```
> J=b11*b22:
```

NON DEGENERATE

Paraboloids. rank(A)=4

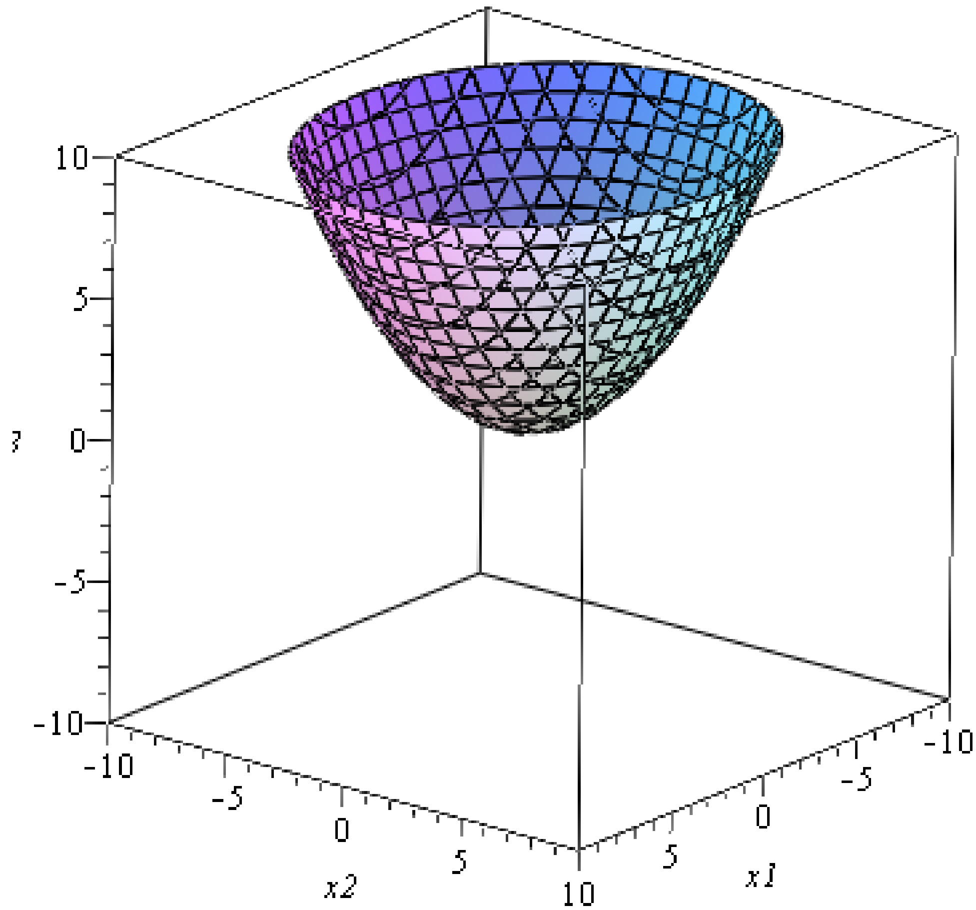
Example of ELLIPTIC PARABOLOID, J>0

```
> b11:=-1/4:b22:=-1/3:
> b03:=1:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := 2\, x3 - \frac{1}{4}\, x1^2 - \frac{1}{3}\, x2^2 \quad (2.1.1)$$

It does not contain lines, it is not a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-10..10,
  numpoints=10000,axes=boxed);
```



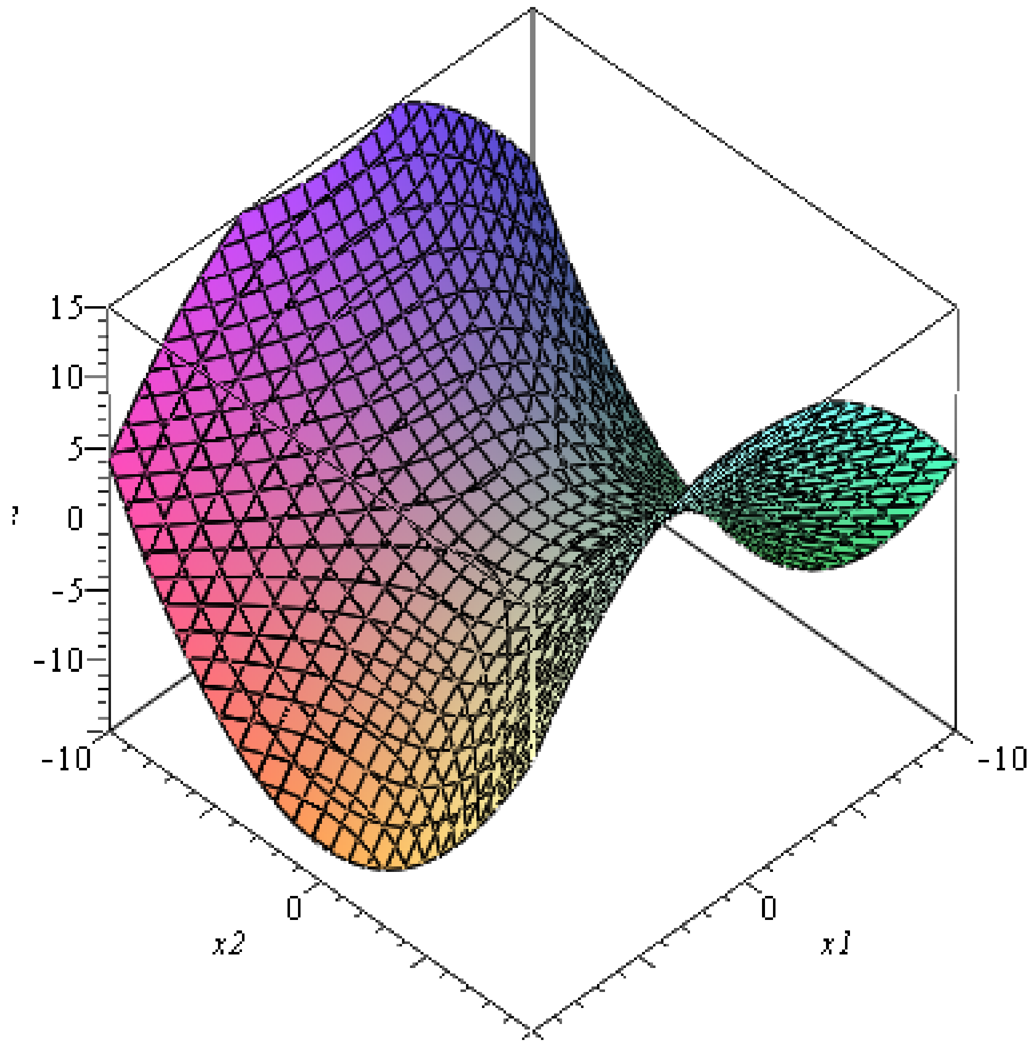
Example of HYPERBOLIC PARABOLOID, J<0

```
> b11:=1/4:b22:=-1/3:
> b03:=1:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := 2\,x3 + \frac{1}{4}\,x1^2 - \frac{1}{3}\,x2^2 \quad (2.1.2)$$

It does contain lines, it is a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-15..15,
  numpoints=10000,axes=boxed);
```



DEGENERATE

```
> restart:with(linalg):
```

```
> J=b11*b22:
```

Cylinders. rank(A)=3

```
> A:=matrix(4,4,[p0,0,0,0,0,b11,0,0,0,0,b22,0,0,0,0,0]);
```

$$A := \begin{bmatrix} p0 & 0 & 0 & 0 \\ 0 & b11 & 0 & 0 \\ 0 & 0 & b22 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.2.1)$$

```
> A00:=submatrix(A,2..4,2..4);
```

$$A00 := \begin{bmatrix} b11 & 0 & 0 \\ 0 & b22 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.2.2)$$

```
> det(A00)=b11*b22*0:
> det(A);
```

$$0 \quad (2.2.3)$$

```
> X:=matrix(1,4,[x0,x1,x2,x3]):
> red_equ:=simplify(evalm(X&*A&*transpose(X)))[1,1];
```

$$red\_equ := x0^2\, p0 + x1^2\, b11 + x2^2\, b22 \quad (2.2.4)$$

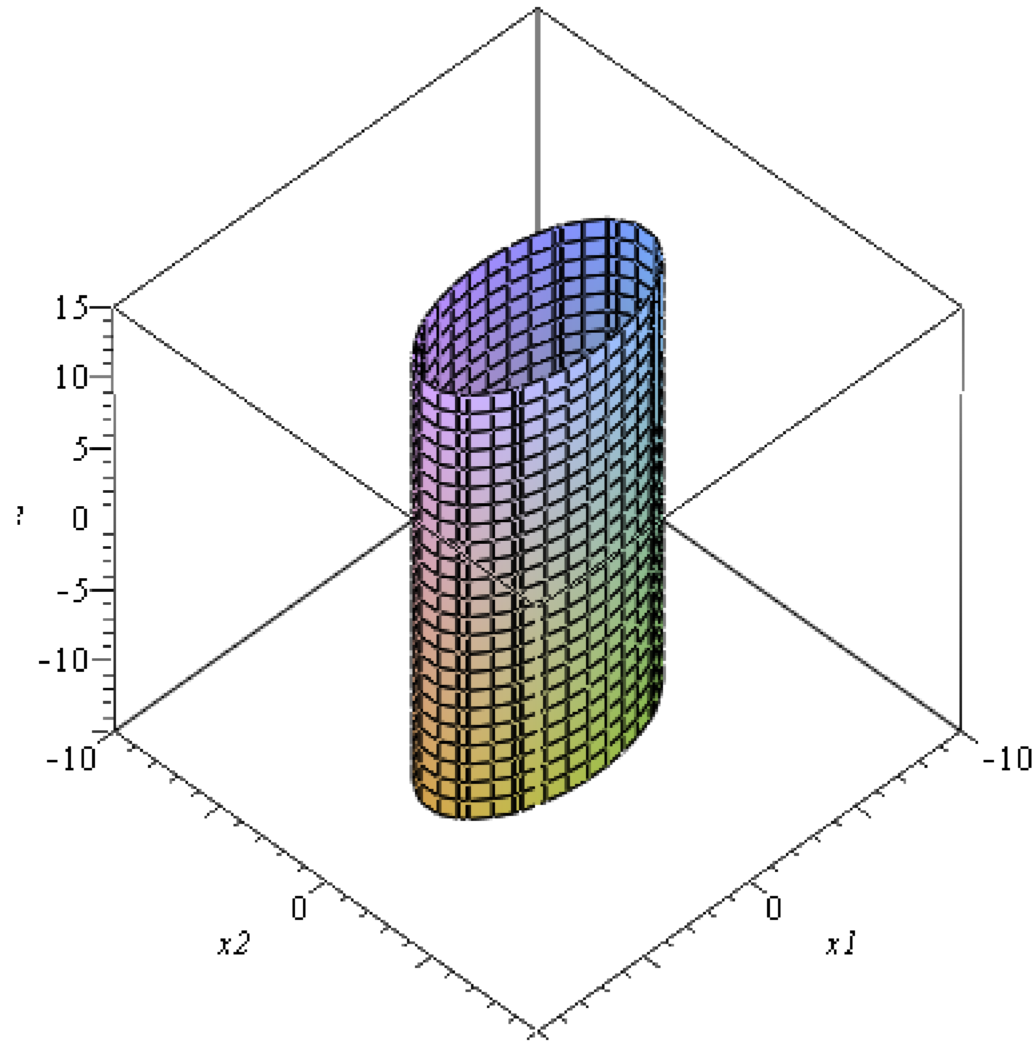
Example of real ELLIPTIC CYLINDER, J>0
(if p0,b11 and b22 have the same sign then it is imaginary)

```
> b11:=1/27:b22:=1/8:
> p0:=-1:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -1 + \frac{1}{27}\, x1^2 + \frac{1}{8}\, x2^2 \quad (2.2.5)$$

It does contain lines, it is a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-15..15,
  numpoints=10000,axes=boxed);
```



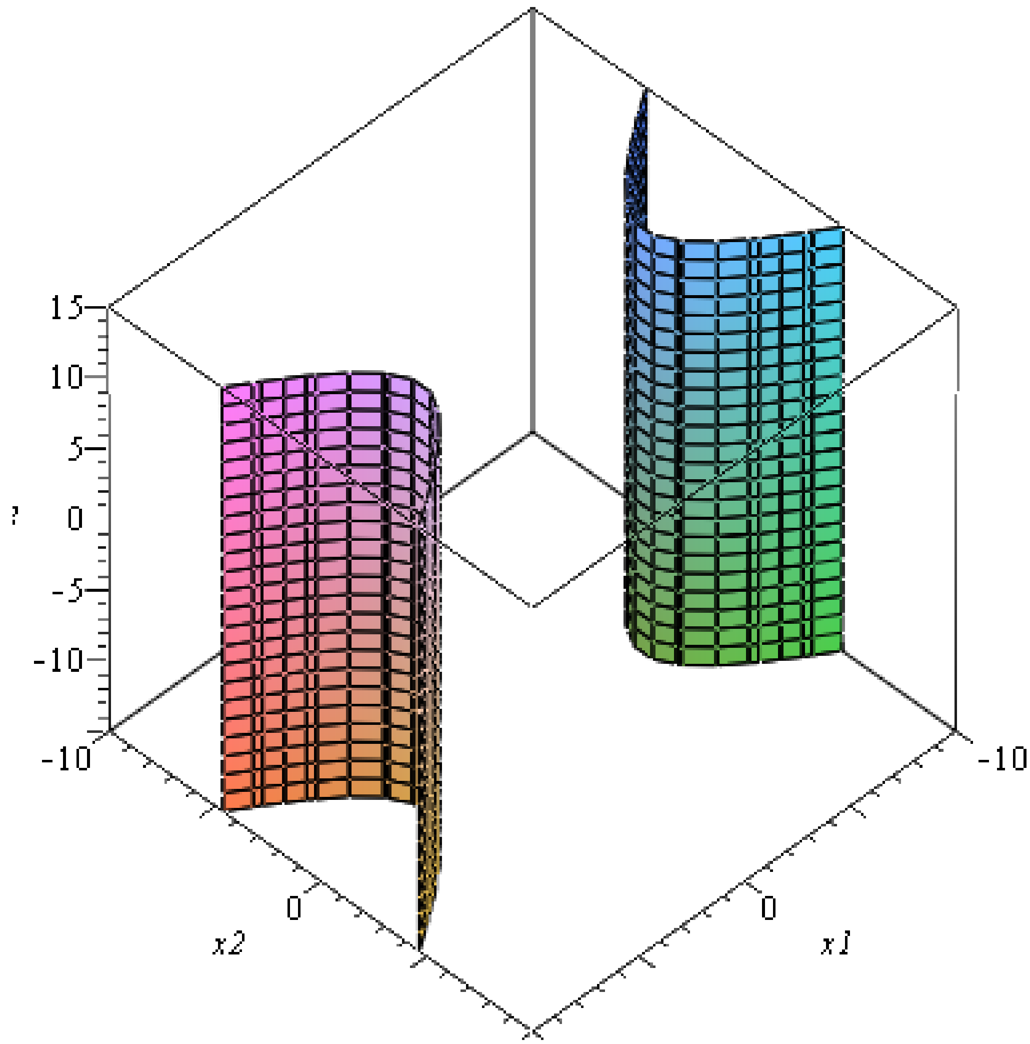
Example of HYPERBOLIC CYLINDER, J<0

```
> b11:=1/27:b22:=-1/8:
> p0:=-1:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -1 + \frac{1}{27}\, x1^2 - \frac{1}{8}\, x2^2 \quad \textbf{(2.2.6)}$$

It does contain lines, it is a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-15..15,
  numpoints=10000,axes=boxed);
```



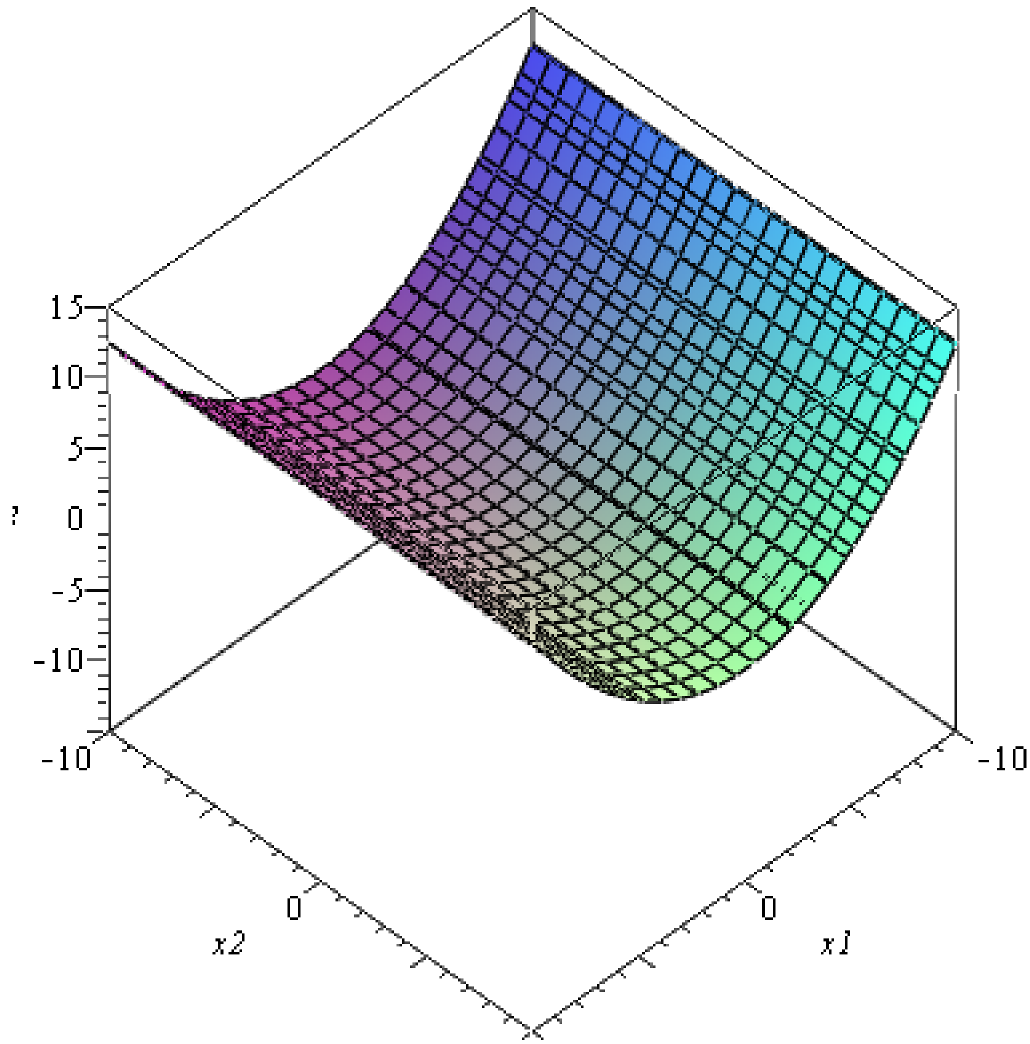
Example of PARABOLIC CYLINDER, J=0

```
> b11:=1/4:b22:=0:
> red_equ := 2*x3*b03*x0+x1^2*b11+x2^2*b22:
> b03:=-1:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -2\,x3 + \frac{1}{4}\,x1^2 \tag{2.2.7}$$

It does contain lines, it is a ruled surface.

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-15..15,
  numpoints=10000,axes=boxed);
```



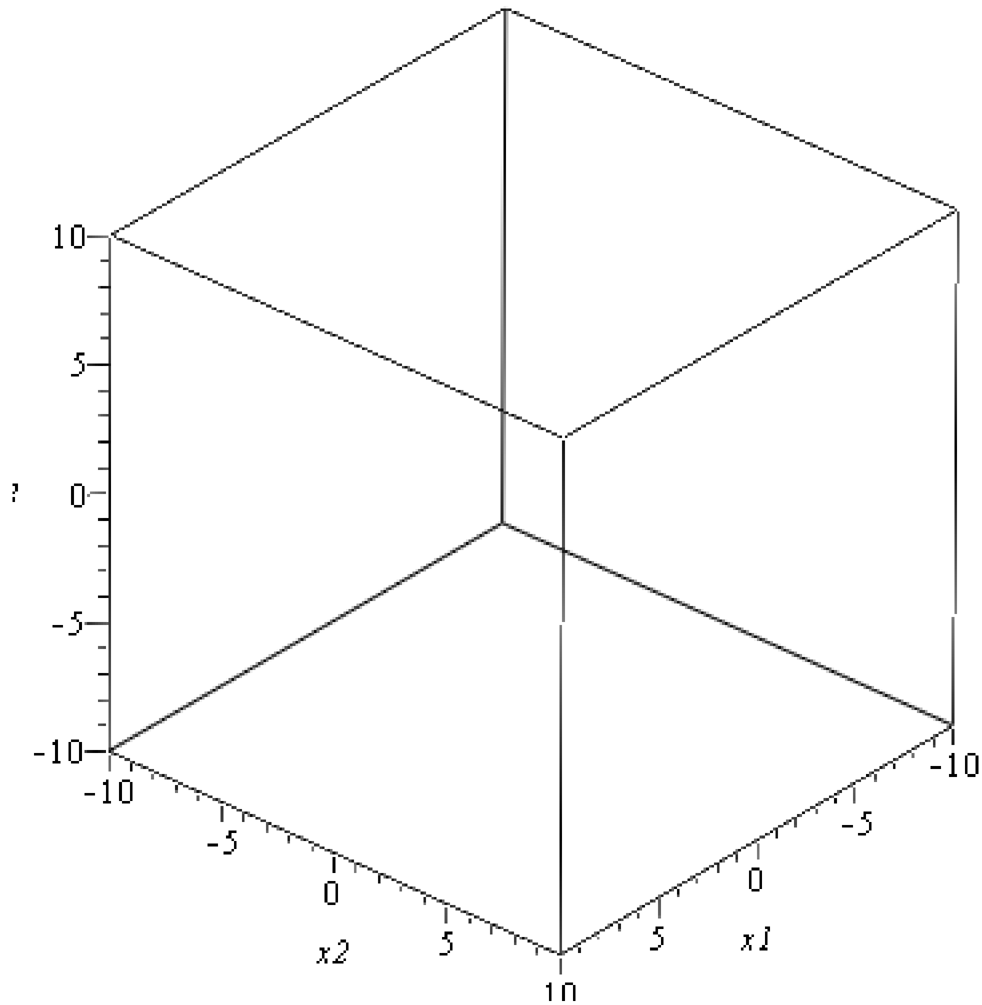
Pair of plains. rank(A)=2

Example of PAIR OF IMAGINARY PLANES (secant in a line). J>0

```
> b11:=1/4:b22:=1/3:
> red_equ:=b11*x1^2+b22*x2^2:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := \frac{1}{4}\, x1^2 + \frac{1}{3}\, x2^2 \qquad \textbf{(2.3.1)}$$

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-10..10,
  numpoints=10000,axes=boxed);
```



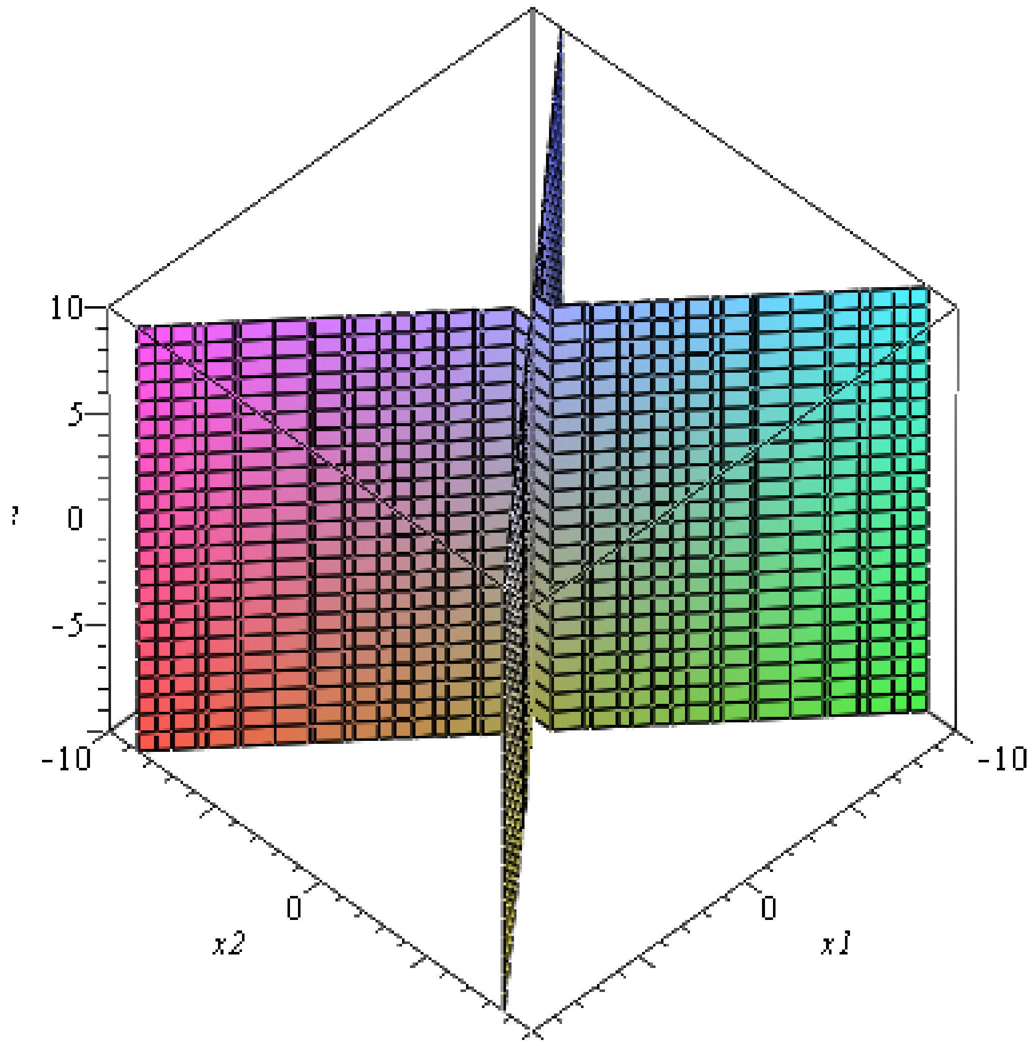

```
> 
```

Example of PAIR OF SECANT REAL PLANES. J<0

```
> b11:=1/4:b22:=-1/3:
> red_equ:=b11*x1^2+b22*x2^2:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := \frac{1}{4}\,x1^2 - \frac{1}{3}\,x2^2 \qquad \textbf{(2.3.2)}$$

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-10..10,
  numpoints=10000,axes=boxed);
```

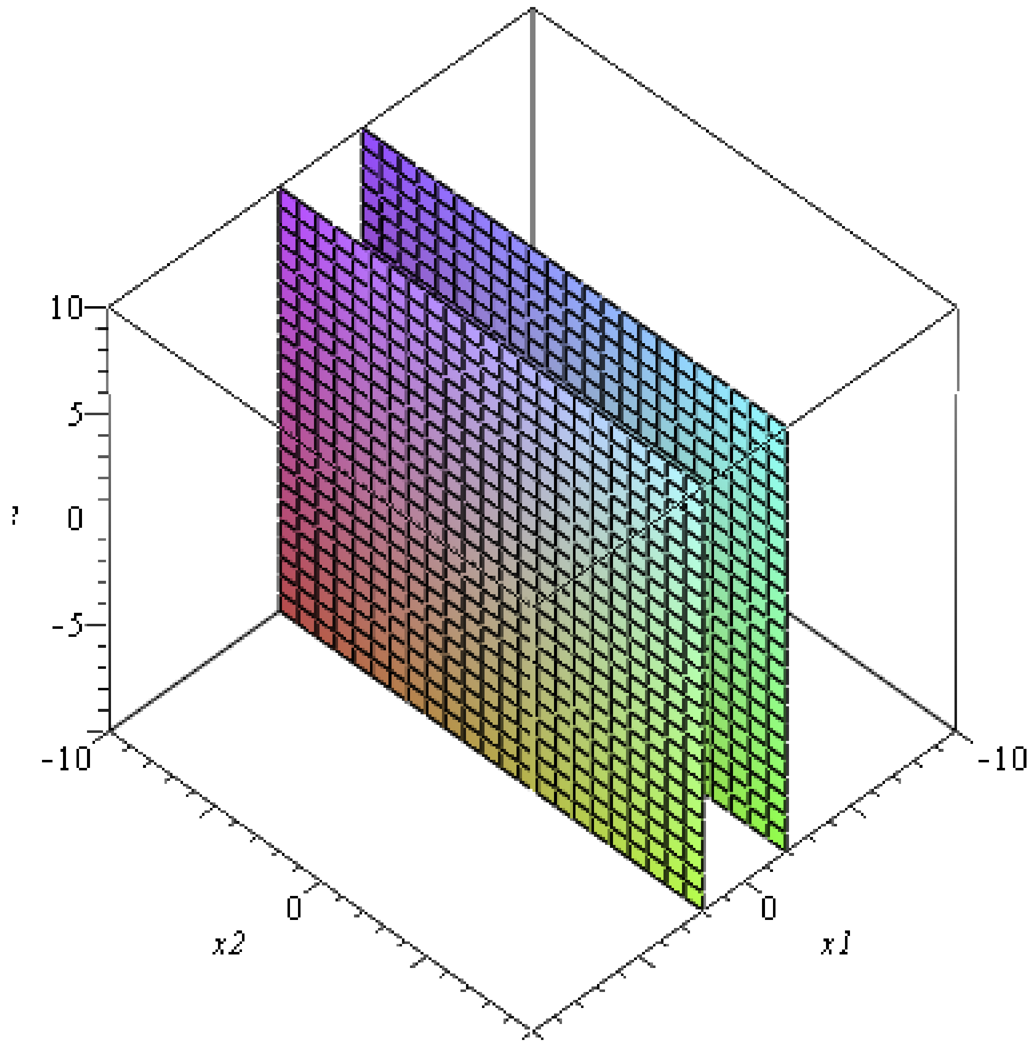


Example of PAIR OF PARALLEL PLANES. J=0

```
> b11:=1/4:b22:=0:
> red_equ:=p0*x0^2+b11*x1^2:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := -1 + \frac{1}{4}\,x1^2 \qquad \textbf{(2.3.3)}$$

```
> with(plots):
> implicitplot3d(red_equ_aff,x1=-10..10,x2=-10..10,x3=-10..10,
  numpoints=10000,axes=boxed);
```

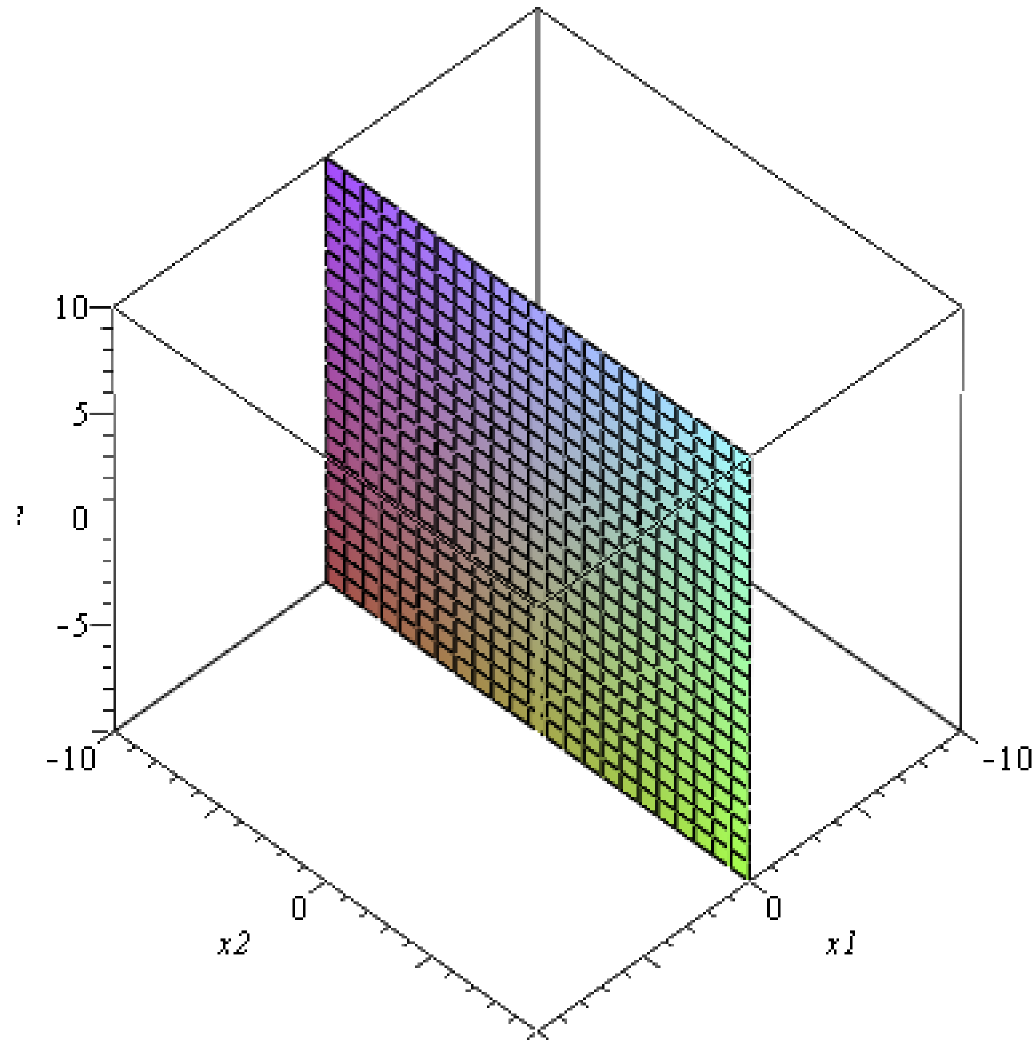


Double plane. rank(A)=1

```
> b11:=1/4:b22:=0:
> red_equ:=b11*x1^2+b22*x2^2:
> red_equ_aff:=subs(x0=1,red_equ);
```

$$red\_equ\_aff := \frac{1}{4}\, x1^2 \qquad (2.4.1)$$

```
> with(plots):
> implicitplot3d(x1=0,x1=-10..10,x2=-10..10,x3=-10..10,
  numpoints=10000,axes=boxed);
```



10
5
0
-5
-10
0
x2
-10
0
x1