KERNEL AND IMAGE OF A LINEAR MAP

Let f:$R^4 \rightarrow R^4$ be the linear map defined by

f(x,y,z,t)=(3x+3y+z+4t, x+3y,3x+2y-z,z+y+4t).

```
> restart; with(linalg):
```

Give the matrix expression of f.

```
> Mf:=matrix(4,4,[2,3,1,4,1,3,0,0,3,2,1,0,1,0,1,4]);
```

$$Mf := \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 3 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \end{bmatrix} \quad \textbf{(1.1.1)}$$

```
> evalm(Mf)&*matrix(4,1,[x,y,z,t])=matrix(4,1,[xp,yp,zp,tp]);
```

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 3 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \end{bmatrix} \&* \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} xp \\ yp \\ zp \\ tp \end{bmatrix} \quad \textbf{(1.1.2)}$$

Obtain the dimension and a basis of Im(f). Im(f)=<f(e1),f(e2),f(e3),f(e4)>

```
> rank(Mf);
```

$$3 \quad \textbf{(1.2.1)}$$

dim Im(f)=3. A basis of Im(f) is

```
> S:=submatrix(Mf,1..4,1..3);
```

$$S := \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \textbf{(1.2.2)}$$

```
> rank(S);
```

$$3 \quad \textbf{(1.2.3)}$$

```
> v1:=[2,1,3,1];v2:=[3,3,2,0];v3:=[1,0,1,1];
```

$$v1 := [2, 1, 3, 1]$$
$$v2 := [3, 3, 2, 0]$$
$$v3 := [1, 0, 1, 1] \quad \textbf{(1.2.4)}$$

{v1,v2,v3} is a basis of Im(f).

Obtain the dimension, a basis of Ker(f) and its cartesian equations.

dim Ker(f)=4-dim Im(f)=1.

```
> zero:=matrix(4,1,[0,0,0,0]);
```

$$zero := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.3.1)$$

```
> linsolve(Mf,zero);
```

$$\begin{bmatrix} 3\,\_t_{1_1} \\ -\_t_{1_1} \\ -7\,\_t_{1_1} \\ \_t_{1_1} \end{bmatrix} \quad (1.3.2)$$

```
> w:=[3,-1,-7,1];
```

$$w := [3, -1, -7, 1] \quad (1.3.3)$$

{w} is a basis of Ker(f).

```
> S2:=submatrix(Mf,1..3,1..4);
```

$$S2 := \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 3 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad (1.3.4)$$

```
> rank(S2);
```

$$3 \quad (1.3.5)$$

```
> equations:=evalm(S2&*matrix(4,1,[x,y,z,t])=matrix(3,1,[0,0,0]
));
```

$$equations := \begin{bmatrix} 2\,x + 3\,y + z + 4\,t \\ x + 3\,y \\ 3\,x + 2\,y + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.3.6)$$

```
> linsolve(S2,matrix(3,1,[0,0,0]));
```

$$\begin{bmatrix} 3\,\_t_{1_1} \\ -\_t_{1_1} \\ -7\,\_t_{1_1} \\ \_t_{1_1} \end{bmatrix} \quad (1.3.7)$$

Is f injective, surjective or bijective? No.

Let f:$R^4 \to R^4$ be the linear map defined by

f(1,0,0,0)=(3,2,1,1), f(0,1,0,0)=(0,1,0,0), f(0,0,1,0)=(3,2,1,4), f(0,0,0,1)=(5,4,0,-1).

```
> restart; with(linalg):
```

Compute $M_f\left(B_4, B_4\right)$.

```
> fe1:=[3,2,1,1];fe2:=[0,1,0,0];fe3:=[3,2,1,4];fe4:=[5,4,0,-1];
```

$$fe1 := [3, 2, 1, 1]$$
$$fe2 := [0, 1, 0, 0]$$
$$fe3 := [3, 2, 1, 4]$$
$$fe4 := [5, 4, 0, -1] \quad \textbf{(2.1.1)}$$

```
> Mf:=transpose(matrix([fe1,fe2,fe3,fe4]));
```

$$Mf := \begin{bmatrix} 3 & 0 & 3 & 5 \\ 2 & 1 & 2 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 4 & -1 \end{bmatrix} \quad \textbf{(2.1.2)}$$

Obtain the dimension and a basis of Im(f).

Im(f)=<f(e1),f(e2),f(e3),f(e4)>

```
> rank(Mf);
```

$$4 \quad \textbf{(2.2.1)}$$

Therefore, Im(f)=R^4.

Obtain the dimension of Ker(f). Is f injective? Is f surjective? Is f bijective?

dim Ker(f)=dim R^4 - dim Im(f)=4-4=0.
Ker(f)={(0,0,0,0)} implies that f is injective and, since f is an endomorphism, f is also surjective.
Finally, we can say that f is bijective.