Computational Logic

Unification and Resolution

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Introduction

[R65], abstract

Theorem-proving on the computer, using procedures based on the fundamental theorem of Herbrand concerning the first-order predicate calculus, is examined with a view towards improving the efficiency and widening the range of practical applicability of these procedures. A close analysis of the process of substitution (of terms for variables), and the process of truth-functional analysis of the result of such substitutions, reveals that both processes can be combined into a single new process (called resolution), iterating which is vastly more efficient than the older cyclic procedures consisting of substitution stages alternating with truth-functional analysis stages.

The theory of the resolution process is presented in the form of a system of first-order logic with just one inference principle (the resolution principle). The completeness of the system is proved; the simplest proof-procedure based on the system is then the direct implementation of the proof of completeness. However, this procedure is quite inefficient, and the paper concludes with a discussion of several principles (called search principles) which are applicable to the design of efficient proof-procedures employing resolution as the basic logical process.

Introduction

From [R65]

- traditionally, a single step in a deduction has been required, for pragmatic and psychological reasons, to be simple enough, broadly speaking, to be apprehended as correct by a human being in a single intellectual act
- from the theoretical point of view, however, an inference principle need only to be *sound* and *effective*
- when the agent carrying out the application of an inference principle is a modern computing machine, [...] more powerful principles [...] become a possibility
- in the system described in this paper, one such inference principle is used. It is called the *resolution principle*, and it is machine-oriented, rather than human-oriented

Introduction

From [R65]

- the main advantage of the resolution principle lies in its ability to allow us to avoid one of the major combinatorial obstacles to efficiency which have plagued earlier theorem-proving procedures
  - (cited in the paper) Gilmore
  - (cited in the paper) Davis-Putnam
  - ground resolution (as presented before)

Substitutions

Formal definition

A *substitution* is a partial function (with finite domain) mapping variables to terms: $\alpha = \{\ x_1/t_1,\ x_2/t_2,\ ..,\ x_n/t_n\ \}$

- $x_1, .., x_n$ are distinct variables
- for every i, x_i does not occur in t_i

Terminology

- *binding*: a pair x_i/t_i
- $Domain(\alpha) = \{\ x \mid x/t \in \alpha\ \}$
- $CoDomain(\alpha) = \{\ y \mid \exists t(\exists x(x/t \in \alpha) \wedge y \text{ occurs in } t)\ \}$
- $\lambda = \{\}$ (*empty substitution*)
- if α is *bijective* from a set V_1 of variables to another set V_2 of variables, then it is called a *renaming*

Substitutions

Examples: variables x, y, z, w

$$\alpha_1 = \{\ x/f(a),\ y/x,\ z/h(b,y),\ w/a\ \}$$

$$Domain(\alpha_1) = \{x,y,z,w\}$$
$$CoDomain(\alpha_1) = \{x,y\}$$

$$\alpha_2 = \{\ x/a,\ y/a,\ z/h(b,c),\ w/f(d)\ \}$$

$$Domain(\alpha_2) = \{x,y,z,w\}$$
$$CoDomain(\alpha_2) = \{\}$$

$$\alpha_3 = \{\ x/y,\ z/w\ \}$$

$$Domain(\alpha_3) = \{x,z\}$$
$$CoDomain(\alpha_3) = \{y,w\}$$

$$\lambda = \{\} = \{\ x/x,\ y/y,\ z/z\ \}$$

Substitutions

Application of α to F

The *application* $F\alpha$ of a substitution α to F is the formula which is obtained by replacing at the same time for all i every occurrence of x_i in F by t_i, for each $x_i/t_i \in \alpha$

$$\alpha = \{\ x/f(a),\ \ y/x,\ \ z/h(b,y),\ \ w/a\ \}$$

- $(p(x,y,z))\alpha = p(f(a), f(a), h(b, f(a))) \rightsquigarrow$ incorrect
- $(p(x,y,z))\alpha = p(f(a), x, h(b, y)) \rightsquigarrow$ correct

Terminology (2)

- F' is an *instance* of F if there exists α such that $F' = F\alpha$
- α is *idempotent* iff $((F\alpha)\alpha = F\alpha)$
  - this happens when $\mathit{Domain}(\alpha) \cap \mathit{CoDomain}(\alpha) = \emptyset$
  - $\{x/a, y/f(b), z/v\}$ is idempotent, $\{x/a, y/f(b), z/x\}$ is not

Substitutions

Composition of substitutions

Given $\alpha = \{x_1/t_1, .., x_n/t_n\}$ and $\beta = \{y_1/s_1, .., y_m/s_m\}$, the *composition* $\alpha\beta$ of these substitutions is defined as:

$$\{\ x_1/(t_1\beta),\ ..,\ x_n/(t_n\beta),\ y_1/s_1,\ ..,\ y_m/s_m\ \}$$

removing the elements such that (1) $x_i \equiv t_i\beta$; or (2) $y_j \in \{x_1, .., x_n\}$

Example

$$\begin{array}{lcl} \alpha & = & \{\ x/3,\ y/f(x,1)\ \} \\ \beta & = & \{\ x/4\ \} \end{array} \qquad \begin{array}{lcl} \alpha\beta & = & \{\ x/3,\ y/f(4,1)\ \} \\ \beta\alpha & = & \{\ x/4,\ y/f(x,1)\ \} \end{array}$$

Properties

$(F\alpha)\beta = F(\alpha\beta)$ $(f.vs.)$

$\alpha\lambda = \lambda\alpha = \alpha$

$(\alpha\beta)\gamma = \alpha(\beta\gamma)$

$\alpha\beta \neq \beta\alpha$

Unifiers

Definition

A substitution α is a *unifier* of two formulæ F and G if $F\alpha = G\alpha$

- in this case, F and G are said to be *unifiable*
- a unifier α of F and G is called *most general unifier* (*MGU*) iff for any other unifier β of F and G there exists γ such that $\beta = \alpha\gamma$

☞ two unifiable formulæ have only one (apart from renaming) *MGU*

Example: $F = p(x, f(x, g(y)), z)$ and $G = p(v, f(v, u), a)$

- $\alpha_1 = \{\ x/v,\ u/g(y),\ z/a\ \}$ $\quad \alpha_2 = \{\ x/a,\ v/a,\ y/b,\ u/g(b),\ z/a\ \}$
- $F\alpha_1 = G\alpha_1 = p(v, f(v, g(y)), a)$
- $F\alpha_2 = G\alpha_2 = p(a, f(a, g(b)), a)$
- α_1 and α_2 are both unifiers, but α_1 is the *MGU*:

  $\alpha_2 = \alpha_1\gamma$ for $\gamma = \{v/a, y/b\}$

Unification Algorithm

Several versions

- Robinson. [R65]. 1965
- Chang, Lee. Symbolic Logic and Mechanical Theorem Proving. 1973
  - a generalization of the presented version
- Martelli, Montanari. An Efficient Unification Algorithm. 1982
- Escalade-Imaz, Ghallab. A Practically Efficient and Almost Linear Unification Algorithm. 1988
- Henckel. An Efficient Linear Unification Algorithm. 1997
- Suciu. Yet Another Efficient Unification Algorithm. 2006
- and many others...

This short list is enough to realize that *efficiency* is the main issue here

Unification Algorithm

Computes the *MGU* of two atoms F and G with the same predicate

$\alpha = \lambda$
while $(F\alpha \neq G\alpha)$
  find the leftmost symbol in $F\alpha$ such that
    the corresponding symbol in $G\alpha$ is different
  let t_F and t_G be the terms in $F\alpha$ and $G\alpha$ which begin with such symbols:
    if (neither t_F nor t_G are variables) or
      (one is a variable which occurs in the other one)
    then FAIL: F and G are not unifiable
    else if (t_F is a variable) **then** $\alpha = \alpha(\{t_F/t_G\})$
    else if (t_G is a variable) **then** $\alpha = \alpha(\{t_G/t_F\})$
α is the *MGU* of F and G

Unification Algorithm

Example: $F = p(x, x)$ and $G = p(f(a), f(b))$

α	$F\alpha$	$G\alpha$	t_F	t_G
λ	$p(x, x)$	$p(f(a), f(b))$	x	$f(a)$
$\{x/f(a)\}$	$p(f(a), f(a))$	$p(f(a), f(b))$	a	b

FAIL: F and G are not unifiable

Example: $F = p(x, f(y))$ and $G = p(z, x)$

α	$F\alpha$	$G\alpha$	t_F	t_G
λ	$p(x, f(y))$	$p(z, x)$	x	z
$\{x/z\}$	$p(z, f(y))$	$p(z, z)$	$f(y)$	z
$\{x/f(y), z/f(y)\}$	$p(f(y), f(y))$	$p(f(y), f(y))$		

F and G have a MGU: $\{\ x/f(y),\ z/f(y)\ \}$

Resolution with Unification

Rule of resolution with unification

Let $C_1 = L_1 \vee D_1$ and $C_2 = \neg L_2 \vee D_2$ two clauses where the atoms L_1 and L_2 have the same predicate symbol. A new clause

$$(D_1\beta \vee D_2)\alpha$$

can be deduced, such that

- β is a renaming such that $C_1\beta$ and C_2 do not have common variables
- α is a unifier of $L_1\beta$ and L_2

The new clause is called the *resolvent* of C_1 and C_2

Resolution with Unification

Rule of factorization

- given a clause $C = L_1 \vee .. \vee L_n \vee D$, where L_i have the same predicate symbol, a new clause $C' = L \vee D\alpha$ can be derived, where
  - α is a unifier (maybe the MGU) of $L_1, .., L_n$
  - $L = L_1\alpha = .. = L_n\alpha$
- L is called a *factor* of $L_1 \vee .. \vee L_n \vee D$

☞ note that the new clause is just an *instance* of the old one, which was obtained by applying α

☞ consequently, the new clause is *less general*, as a logical fact, than the old one

☞ in other words, by factorizing, we forget about a part of the information

Resolution with Unification

Resolution with Unification (RU) step

Possibly apply the rule of factorization, followed by resolution with unification

- in the system described in this paper, one such inference principle is used. It is called the *resolution principle*, and it is machine-oriented, rather than human-oriented [R65]

☞ note that factorization is not compulsory: the new resolvent can be obtained with or without factorizing

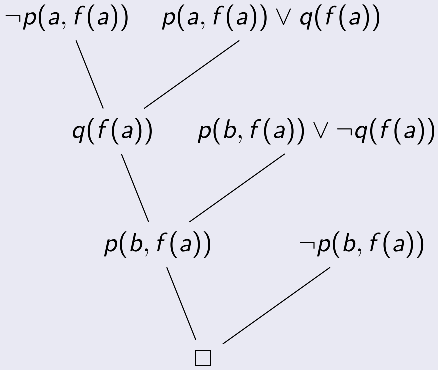
The method

It is possible to build resolution trees where the resolvent of each two clauses can be obtained by means of an RU step

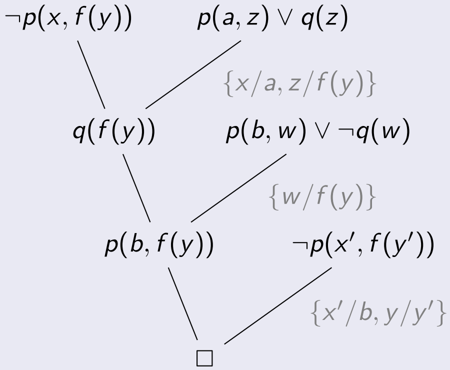
- for every step of *ground resolution*, there is a step of *resolution with unification*

Resolution with Unification

$C_1 = \neg p(x, f(y)),\ C_2 = p(a, z) \vee q(z),\ C_3 = p(b, u) \vee \neg q(u)$



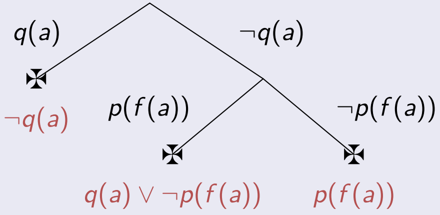
ground instance resolution

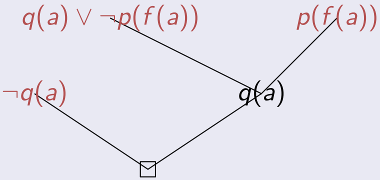


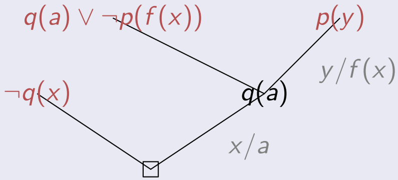
resolution with unification

Semantic trees vs. resolution trees

$$
\begin{aligned}
C_1 &= p(y) \\
C_2 &= q(a) \vee \neg p(f(x)) \\
C_3 &= \neg q(x)
\end{aligned}
$$







Resolution with Unification

Lemma (Lifting Lemma)

Let C_1 (resp., C_2) be a clause and B_1 (resp., B_2) one of its ground instances. If B is a resolvent of B_1 and B_2, then

- *there exists a clause C which has B as one of its ground instances*
- *C results from a resolution step on C_1 and C_2 w.r.t. a literal L which is a common factor of C_1 and C_2:*

$$C_1 = L_1 \vee .. \vee L_n \vee D_1 \qquad\qquad C_2 = \neg L_{n+1} \vee .. \vee \neg L_{n+m} \vee D_2$$
$$C = (D_1\rho \vee D_2)\theta$$

- *ρ is a renaming such that $C_1\rho$ and C_2 have no common variables*
- *θ is an MGU: $L_1\rho\theta = .. = L_n\rho\theta = L_{n+1}\theta = .. = L_{n+m}\theta = L$*

Resolution with Unification

MGU resolution rule

☞ special case of resolution with unification, where the unifier is the *MGU*

Let

$$C_1 = L_1 \vee .. \vee L_n \vee D_1 \qquad\qquad C_2 = \neg L_{n+1} \vee .. \vee \neg L_{n+m} \vee D_2$$

where all L literals have the same predicate symbol. A new clause

$$(D_1\rho \vee D_2)\theta$$

can be deduced, where

- ρ is a renaming such that $Vars(C_1\rho) \cap Vars(C_2) = \emptyset$
- θ is the *MGU* of $L_1\rho, \ldots, L_n\rho, L_{n+1}, \ldots, L_{n+m}$

Resolution with Unification

Lemma (*MGU* resolution rule, correctness)

$[\ \forall x_1..x_p C_1,\quad \forall y_1..y_q C_2\ ] \quad \vdash \quad \forall z_1..z_r((D_1\rho \vee D_2)\theta)$ *is correct, where*

- $\{x_1, .., x_p\} = Vars(C_1)$, $\{y_1, .., y_q\} = Vars(C_2)$, $\{z_1, .., z_r\} = Vars((D_1\rho \vee D_2)\theta)$
- *ρ is a renaming of $x_1..x_p$ defined as above*
- $\theta = MGU(L_1\rho, \ldots, L_n\rho, L_{n+1}, \ldots, L_{n+m})$

Proof.

❶	$\forall x_1..x_p(L_1 \vee .. \vee L_n \vee D_1)$	hypothesis ($C_1 = \overline{L} \vee D_1$)
❷	$\forall z_1..z_r(\neg L_{n+1} \vee .. \vee \neg L_{n+m} \vee D_2)$	hypothesis ($C_2 = \overline{\neg L} \vee D_2$)
❸	$F \vee E_1$	apply ρ and θ to C_1, idempotence $F \vee .. \vee F = F$
❹	$\neg F \vee E_2$	apply θ to C_2, idempotence $\neg F \vee .. \vee \neg F = \neg F$
❺	$E_1 \vee E_2$	cut on ❸ and ❹
❻	$\forall z_1..z_r((D_1\rho \vee D_2)\theta)$	generalization of ❺

An important observation

☞ the rule of *MGU* resolution does *not* imply that all possible factorization steps have been performed

- actually, factorization helps in some cases, but may make the problem unsolvable in other cases

$$
\begin{array}{ll}
C_1 & \neg p(x,y,u) \vee \neg p(y,z,v) \vee \neg p(u,z,w) \vee p(x,v,w) \\
C_2 & \neg p(x,y,u) \vee \neg p(v,z,y) \vee \neg p(x,v,w) \vee p(u,z,w) \\
C_3 & p(x,e,x) \\
C_4 & p(x,i(x),e) \\
C_5 & p(i(x),x,e) \\
C_6 & \neg p(c,c,e)
\end{array}
$$

Resolution with Unification

An important observation

☞ the rule of *MGU* resolution does *not* imply that all possible factorization steps have been performed

- actually, factorization helps in some cases, but may make the problem unsolvable in other cases

$$
\begin{array}{ll}
C_1 & \neg p(x,y) \vee \neg q(f(y),x) \\
C_2 & p(x,g(x)) \\
C_3 & r(x,a) \vee \neg p(b,g(x)) \vee r(z,z) \\
C_4 & q(f(g(x)),a) \vee \neg r(x,a) \vee \neg r(a,y) \\
C_5 & p(x,g(y))
\end{array}
$$

Resolution with Unification

Lemma

Let $\mathcal{C}$ be an unsatisfiable set of clauses with a closed semantic tree of depth $n \geq 1$. Then there is a set R of resolvents of $\mathcal{C}$ such that $\mathcal{C}' = \mathcal{C} \cup R$ has a closed semantic tree of depth $n - 1$

Proof.

1. let B_1, B_2 be two ground instances of $C_1, C_2 \in \mathcal{C}$ which are false in two failure nodes (brothers) at level n (the deepest in the tree)
2. the resolvent of B of B_1 and B_2 is false in the parent node (depth $n - 1$)
3. by the Lifting Lemma, there exists an *MGU* resolvent C of C_1 and C_2 such that B is a ground instance of C
4. let R be the set of such Cs, obtained by considering all pairs of failure nodes at the maximum depth n
5. a closed semantic tree of $\mathcal{C} \cup R$ can be constructed which has maximum depth $n - 1$ (essentially, by pruning the initial tree)

Resolution with Unification

Lemma (*MGU* resolution)

If a set $\mathcal{C}$ of clauses is unsatisfiable, then $\Box$ is deduced from it by MGU resolution

Proof.

1. $UNSAT(\mathcal{C})$
2. there exists an n-deep closed semantic tree (by Herbrand's Theorem)
3. if $n = 1$, then
   - B_1 and B_2, ground instances of C_1 and C_2, are false at the only non-root nodes
   - their resolvent must be false in the root, so that it must be $\Box$
   - $\Box$ is also the resolvent of C_1 and C_2 (by Lifting Lemma, and since $\Box$ is ground)
4. if $n > 1$, then
   - there exists a set R of resolvents of $\mathcal{C}$ clauses, such that $\mathcal{C}' = \mathcal{C} \cup R$ has a $(n-1)$-deep closed semantic tree (by the Lemma above)
   - the rest follows by induction

Resolution with Unification

Theorem (*MGU* resolution)

A set $\mathcal{C}$ of clauses is unsatisfiable iff $\Box$ can be deduced from it by MGU resolution ($\mathcal{C} \vdash_{MGU} \Box$)

Proof ($\longrightarrow$).

Follows by the *MGU* resolution Lemma

Proof ($\longleftarrow$).

1. $\mathcal{C} \vdash \Box$ by *MGU* resolution
2. $\mathcal{C} \models \Box$ for the correctness of *MGU* resolution
3. $\Box$ is false in every interpretation
4. $\mathcal{C}$ must be false in every interpretation
5. $UNSAT(\mathcal{C})$

Method of Saturation

Let $\mathcal{C}$ be a set of clauses

$S_0 = \mathcal{C}$
$n = 0$
repeat
   if $(\square \in S_n)$ **then STOP**: $UNSAT(\mathcal{C})$
   else
      $S_{n+1} = \{\text{resolvents of } C_1 \text{ and } C_2 \mid C_1 \in S_1 \cup .. \cup S_n,\ C_2 \in S_n\}$
      if $(S_{n+1} = \emptyset)$ or $(S_{n+1} \subseteq S_1 \cup .. \cup S_n)$ **then STOP**: $SAT(\mathcal{C})$
      $n = n + 1$

Completeness: $UNSAT(\mathcal{C})$ iff $\square$ is derived

- the construction of S_{n+1} requires considering all possible factors of C_1 and C_2
- this method generates *all* and *only* the resolvents of $\mathcal{C}$ clauses
- a number of redundant clauses are generated

Method of Saturation

Example: $\mathcal{C} = \{p \vee q,\ \neg p \vee q,\ p \vee \neg q,\ \neg p \vee \neg q\}$

$$S_0 = \begin{array}{ll} (1) & p \vee q \\ (2) & \neg p \vee q \\ (3) & p \vee \neg q \\ (4) & \neg p \vee \neg q \end{array} \qquad S_1 = \begin{array}{lll} (5) & q & (1,2) \\ (6) & p & (1,3) \\ (7) & q \vee \neg q & (1,4) \\ (8) & p \vee \neg p & (1,4) \\ (9) & q \vee \neg q & (2,3) \\ (10) & p \vee \neg p & (2,3) \\ (11) & \neg p & (2,4) \\ (12) & \neg q & (3,4) \end{array}$$

even after one step there are redundant and tautological clauses

Method of Saturation

Conclusion

- *MGU* resolution allows to decide satisfiability without the need to use ground instances
- however, saturation is not efficient since it generates many useless clauses
  - the raw implementation of the Resolution Principle would produce a very inefficient refutation procedure [R65]
  - by Church's Theorem we know that for some inputs S this procedure, and in general all correct refutation procedures, will not terminate [R65]

Example [R65]

$$C_1 = q(a) \qquad\qquad C_2 = \neg q(x) \vee q(f(x))$$

at each step, $q(f^n(a))$ is generated, for n increasing by 1 each time