Computational Logic

Resolution Strategies

Damiano Zanardini

UPM European Master in Computational Logic (EMCL)
School of Computer Science
Technical University of Madrid
damiano@fi.upm.es

Academic Year 2009/2010

Introduction

The problem

- the method of *saturation* from a set $\mathcal{C}$ generates, if not limited, a big number of clauses which are redundant or irrelevant
- it is necessary to use systematic *selection rules* which make the process *simpler* and *computationally efficient*
- two kinds of criteria
  - *simplification strategies*: reducing the number of clauses
  - *refinement strategies*: limiting the generation of clauses

Terminology

- $\mathcal{C}$ is the *initial* set of clauses
- $\mathcal{C}'$ is the *current* set of clauses (at some point during the deduction process where we want to apply the rules)

Simplification Strategies

1. Elimination of identical clauses

- obviously, $\mathcal{C} \vdash_{MGU} \Box$ iff $\Box$ can be derived by eliminating identical clauses (apart from one copy, of course)

How to do it

- if a clause is generated which already appears in $\mathcal{C}'$, then it is not included

☞ note that, in a computer algorithm, this is a check which may involve comparing the new clause with the entire set $\mathcal{C}'$ of existing clauses

Simplification Strategies

2. Elimination of clauses with pure literals

- a literal L is *pure* iff there does not exist in the set a literal $\neg L'$ where L and L' are unifiable
- $\mathcal{C} \vdash_{MGU} \Box$ iff $\Box$ can be derived after removing from $\mathcal{C}$ clauses with pure literals
  - a clause with pure literals is useless for refutation since it will never be eliminated by resolution

How to do it

- clauses with pure literals are removed from the set
- it is enough to apply this strategy *once*, since no new clauses with pure literals will be generated

Simplification Strategies

3. Elimination of tautological clauses
   - $\mathcal{C} \vdash_{MGU} \Box$ iff $\Box$ can be derived from $\mathcal{C}$ after removing tautologies

How to do it

- if a clause is generated which is a tautology, then it is not included in $\mathcal{C}'$

Simplification Strategies

Example: $\mathcal{C} = \{p \vee q,\ \neg p \vee q,\ p \vee \neg q,\ \neg p \vee \neg q\}$

- by applying all the simplification rules, the derivation comes to be

$$
\begin{array}{lll}
(1) & p \vee q & \\
(2) & \neg p \vee q & \\
(3) & p \vee \neg q & \\
(4) & \neg p \vee \neg q & \\
(5) & q & (1,2) \\
(6) & p & (1,3) \\
(7) & \neg p & (2,4) \\
(8) & \neg q & (3,4) \\
(9) & \square & (5,8)
\end{array}
$$

- it must be noted that no other *smart* strategy has been used

	pairs considered	resolvents generated
first iteration	6 (6)	8 (8)
second iteration	19 (60)	9 (....)

Simplification Strategies

4. Elimination of subsumed clauses

- a clauses C *subsumes* another clause D if there exists a substitution α such that $C\alpha$ is a subformula of D: $D = C\alpha \vee D'$

Example

$D = p(f(a), x) \vee q(g(y), y) \vee r(b)$ is subsumed by $C = r(z) \vee p(f(u), v)$ under $\alpha = \{u/a, v/x, z/b\}$

Simplification Strategies

Lemma (subsumed clauses)

The set $\{C_1, .., C_n, C, C\alpha \vee D\}$ *is unsatisfiable iff* $\{C_1, .., C_n, C\}$ *is*

Proof ($\rightarrow$).

1. $UNSAT(\{C_1, .., C_n, C, C\alpha \vee D\})$
2. suppose $SAT(\{C_1, .., C_n, C\})$: there exists a Herbrand interpretation I_H which makes all C_i and C true
3. I_H makes $C\alpha$ true (since universal quantification is implicit), so that it also makes $C\alpha \vee D$ true
4. I_H satisfies $\{C_1, .., C_n, C, C\alpha \vee D\}$: contradiction with ❶
5. $UNSAT(\{C_1, .., C_n, C\})$

Simplification Strategies

Lemma (subsumed clauses)

The set $\{C_1, .., C_n, C, C\alpha \vee D\}$ *is unsatisfiable iff* $\{C_1, .., C_n, C\}$ *is*

Proof ($\leftarrow$).

1. $UNSAT(\{C_1, .., C_n, C\})$
2. there is no interpretation which makes C_i and C true
3. there is no interpretation which makes C_i, C and $C\alpha \vee D$ true
4. $UNSAT(\{C_1, .., C_n, C, C\alpha \vee D\})$

Simplification Strategies

Procedure for deciding subsumption: is C_1 subsumed by C_2?

Procedure $\textsc{is\_subsumed\_by}\,(C_1, C_2)$:

---

if (C_2 is empty) **then return** YES: C_1 is subsumed by C_2 **else**
  if for some p $((p(\bar{t}) \in C_2$ and there is no $p(\bar{t}') \in C_1) \vee$
  $(\neg p(\bar{t}) \in C_2$ and there is no $\neg p(\bar{t}') \in C_1))$
  then return NO: C_1 is not subsumed by C_2
  $L_2 = q(\bar{t})$ is the first literal in C_2
  $CL_1 = \{q(\bar{t'}) \in C_1 \mid \bar{t}'$ are terms$\}$
  for each $(L \in CL_1)$
    $\mu_L = MGU(L_2, L)$ such that $Domain\,(\mu_L) \cap Vars(L) = \emptyset$
    if (such μ_L exists)
      C_2' is C_2 where L_2 has been removed
      $C_2'' = C_2' \mu_L$
      if $(\textsc{is\_subsumed\_by}\,(C_1, C_2'') =$ YES$)$ **then**
        return YES: C_1 is subsumed by C_2
  return NO: C_1 is not subsumed by C_2

Simplification Strategies

Example: $C_1 = p(a, s) \vee p(b, z) \vee \neg q(f(z), b)$
$C_2 = p(x, y) \vee \neg q(w, x)$

- $L_2 = p(x, y)$
- $CL_1 = \{p(a, s), p(b, z)\}$
- $\mu_{p(a,s)} = \{x/a, y/s\}$
- $\mu_{p(b,z)} = \{x/b, y/z\}$
- $\mu_{p(a,s)} \rightsquigarrow C_2'' = \neg q(w, a)$
- $q(w, a)$ and $q(f(z), b)$ are not unifiable
- $\mu_{p(b,z)} \rightsquigarrow C_2'' = \neg q(w, b)$
- and $MGU(q(w, b), q(f(z), b)) = \{w/f(z)\}$
- therefore, C_1 is subsumed by C_2

Simplification Strategies

Example: $C_1 = p(a,s) \vee p(b,z) \vee \neg q(f(z),b)$
$C_2 = p(x,y) \vee \neg q(w,x)$

	C_2	L_2	L (one from CL_1)	μ_L
1	$p(x,y) \vee \neg q(w,x)$	$p(x,y)$	$p(a,s)$	$\{x/a, y/s\}$
2	$\neg q(w,a)$	$\neg q(w,a)$	$\neg q(f(z),b)$	fail
1	$p(x,y) \vee \neg q(w,x)$	$p(x,y)$	$p(b,z)$	$\{x/b, y/z\}$
2	$\neg q(w,b)$	$\neg q(w,b)$	$\neg q(f(z),b)$	$\{w/f(x)\}$
3	$\square$: YES			

Search Trees

Derivations

A *derivation* of C from $\{C_1, .., C_n\}$ is a sequence $\langle C_1, .., C_n, R_1, .., R_m \rangle$ such that

- every R_i is the resolvent of two previous clauses
- no resolution step is done more than once
- $R_m = C$

Refutations

A *refutation* of $\{C_1, .., C_n\}$ is a derivation of $\square$ from $\{C_1, .., C_n\}$

Facts

- a derivation is a correct deduction (by correctness of *MGU* resolution)
- if $UNSAT(\mathcal{C})$, then there exists a refutation for $\mathcal{C}$ (by completeness of *MGU* resolution)

Search Trees

Search tree T of $\{C_1, .., C_n\}$

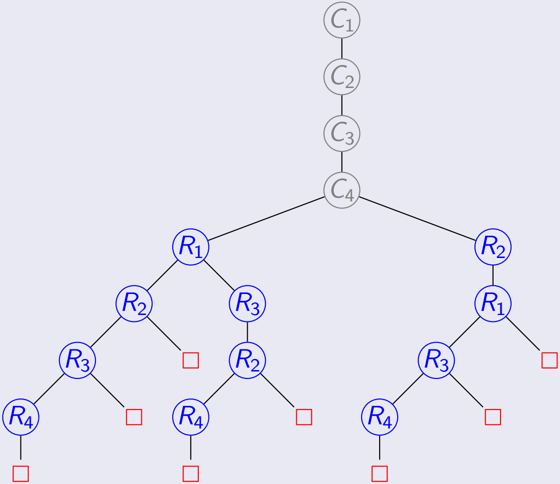
- C_1 is the root of T
- C_{i+1} is a node of T, where C_i is its (direct) predecessor $(1 \leq i < n)$
- let N_p be the set of predecessors of the node N, plus N itself
- every node N of level $i \geq n$ has, as successors, *all* clauses R such that
  - R is a resolvent of two clauses belonging to N_p
  - $R \notin N_p$

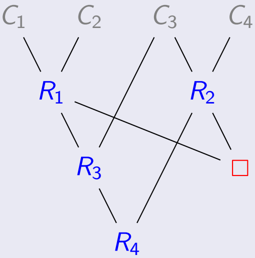
Properties

- every path from C_1 to a node N is a derivation of N
- every possible derivation is represented by a path in the search tree
- the tree for $\mathcal{C}$ contains all the resolvents for $\mathcal{C}$
- if $\square$ is a resolvent, then there is at least a node labeled with $\square$

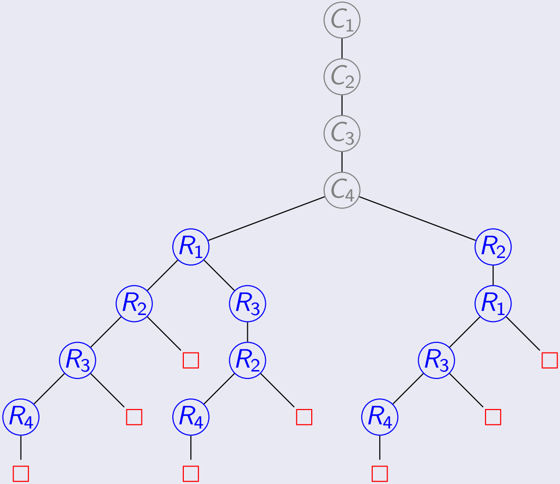
Search Trees

Derivations and search trees





Derivations and search trees



$C_1\, C_2\, C_3\, C_4\, R_1\, R_2\, R_3\, R_4\, \square$
$C_1\, C_2\, C_3\, C_4\, R_1\, R_2\, R_3\, \square$
$C_1\, C_2\, C_3\, C_4\, R_1\, R_2\, \square$
$C_1\, C_2\, C_3\, C_4\, R_1\, R_3\, R_2\, R_4\, \square$
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$C_1\, C_2\, C_3\, C_4\, R_2\, R_1\, R_3\, \square$
$C_1\, C_2\, C_3\, C_4\, R_2\, R_1\, \square$

Search Trees

Restricted search trees

- refinement strategies make the search simpler by only considering derivations which satisfy a given property
  - i.e., trees with a given shape
- a search tree can be reduced by imposing conditions on the successors of a node N, by restricting the clauses D_i and D_j which can produce resolvents starting from N

Linear Resolution

Linear derivations

A linear derivation of C_m from $\{C_1, .., C_n\}$ is a sequence

$$\langle C_1, .., C_n, C_{n+1}, .., C_m \rangle$$

such that

- C_{n+1} is the resolvent of two clauses of $\{C_1, .., C_n\}$ (*header clauses*)
- for every $i > n + 1$, C_i is the resolvent of C_{i-1} and another clause C_j, with $j < i - 1$

Linear Resolution

Properties

Linear resolution is *complete*: $UNSAT(\mathcal{C})$ iff there exists a linear refutation of $\mathcal{C}$

- derivations can be restricted to linear derivations
- search trees can be restricted to linear search trees

☞ what's wrong the the search tree and the resolution tree above?

In a derivation of C from $\mathcal{C}$, it is not necessary to try all the clauses in $\mathcal{C}$ as a starting point for the refutation (of $\neg C$)

- if a set $\mathcal{C}$ is satisfiable and $\mathcal{C} \cup \neg C$ is not, then there exists a linear refutation starting from $\neg C$

Input Resolution

Input derivations

An input derivation of C_m from $\{C_1, .., C_n\}$ is a sequence

$$\langle C_1, .., C_n, C_{n+1}, .., C_m \rangle$$

such that

- for every $i > n$, C_i is a resolvent of $C_k \in \{C_1, .., C_n\}$ and another C_j $(j < i)$

Example

$$C_1 = \neg p(x) \vee q(x) \qquad C_2 = \neg r(x) \vee \neg q(x) \qquad C_3 = r(a)$$
$$C_4 = s(a), \qquad C_5 = \neg s(x) \vee p(x)$$

- input refutation from C_1:
$$\begin{array}{ll} R_1 = \neg p(x) \vee \neg r(x) & (C_1, C_2) \\ R_2 = \neg s(x) \vee \neg r(x) & (R_1, C_5) \\ R_3 = \neg s(a) & (R_2, C_3) \\ R_4 = \Box & (R_3, C_4) \end{array}$$

Input Resolution

Input derivations

An input derivation of C_m from $\{C_1, .., C_n\}$ is a sequence

$$\langle C_1, .., C_n, C_{n+1}, .., C_m \rangle$$

such that

- for every $i > n$, C_i is a resolvent of $C_k \in \{C_1, .., C_n\}$ and another C_j ($j < i$)

Example

$$C_1 = \neg p(x) \vee q(x) \qquad C_2 = \neg r(x) \vee \neg q(x) \qquad C_3 = r(a)$$
$$C_4 = s(a), \qquad C_5 = \neg s(x) \vee p(x)$$

- input refutation from C_5:
$$\begin{array}{ll} R_1 = p(a) & (C_4, C_5) \\ R_2 = q(a) & (R_1, C_1) \\ R_3 = \neg r(a) & (R_2, C_2) \\ R_4 = \Box & (R_3, C_3) \end{array}$$

Input Resolution

Example: $C_1 = p \vee q$, $C_2 = \neg q$, $C_3 = r \vee q$, $C_4 = \neg r$

- *input non-linear* refutation from C_1:
  $R_1 = p \quad (C_1, C_2)$
  $R_2 = r \quad (C_2, C_3)$
  $R_3 = \square \quad (R_2, C_4)$
- since R_1 is not involved in the rest of the derivation, we can build an *input linear* refutation from the first one:
  $R_1 = r \quad (C_2, C_3)$
  $R_2 = \square \quad (R_1, C_4)$

Input Resolution

Lemma

Given an input non-linear derivation of R_m, it is possible to construct an input linear derivation of R_m

Proof.

Let $C_1, .., C_n, R_1, .., R_m$ an input non-linear derivation of R_m

1. let R_{k+1} $(n+1 \leq k \leq m)$ the first resolvent which is non-linearly derivated
2. R_{k+1} is the resolvent of $C \in \{C_1, .., C_n\}$ and R_j $(1 \leq j < k)$
3. for input resolution, R_{k+1} and R_k cannot resolve with each other
4. for ❸, it is possible to generate two independent derivations
    - $C_1, .., C_n, R_1, .., R_k, ..$ (linear until R_k)
    - $C_1, .., C_n, R_1, .., R_j, R_{k+1}, ..$ (linear until R_{k+1})
5. one of these derivations will terminate in R_m

☞ we can *linearize* such derivation by further applying the lemma to it

Input Resolution

(counter)Ex.: $C_1 = p \vee q,\ C_2 = \neg p \vee q,\ C_3 = r \vee \neg q,\ C_4 = \neg r \vee \neg q$

- *non-input non-linear*:

  $R_1 = q \vee q \quad (C_1, C_2)$

  $R_2 = \neg q \vee \neg q \quad (C_3, C_4)$

  $R_3 = \Box \quad (R_1, R_2)$

- for every non-linear derivation there exists a linear equivalent one:

  $R_1 = q \vee q \quad (C_1, C_2)$

  $R_2 = r \quad (R_1, C_3)$

  $R_3 = \neg q \quad (R_2, C_4)$

  $R_4 = \Box \quad (R_3, R_1)$

- is it possible to find an input derivation for every non-input derivation?

Input Resolution

Input resolution is *not* complete

It is *not* possible to say that, for every unsatisfiable set of clauses, there exists an input refutation

Ex.
$$
\begin{array}{l}
p \vee q \\
\neg p \vee r \\
p \vee \neg q \\
s \vee q \\
s \vee \neg q \\
\neg s \vee \neg r
\end{array}
$$

Directed Resolution (Wos-Robinson-Carson, 1965)

Directed derivations

A directed derivation of C_m from $\{C_1, .., C_n\}$, with a *support set* $S \subset \mathcal{C}$, is a sequence $\langle C_1, .., C_n, C_{n+1}, .., C_m \rangle$ such that

- for every $i > n$, C_i is a resolvent of two previous clauses in the sequence, such that at least one of them does *not* belong to S
- clauses in S are *support clauses*, while clauses in $\mathcal{C} \setminus S$ are *goal clauses*
- this technique is motivated by the fact that:
  - suppose we want to prove B from $A_1 \wedge .. \wedge A_k$
  - i.e., that $A_1 \wedge .. \wedge A_k \wedge \neg B$ is unsatisfiable
  - in this case, $A_1 \wedge .. \wedge A_k$ is usually satisfiable in itself
  - therefore, it might be wise to avoid resolving two clauses of such set
  - the support set identifies the subset of $\mathcal{C}$ which is supposed to be satisfiable (the result to be proven is not in the support set)

Directed Resolution (Wos-Robinson-Carson, 1965)

Example

$$\mathcal{C} = \{C_1 = s \vee t,\ \ C_2 = \neg s \vee p,\ \ C_3 = \neg q \vee r,\ \ C_4 = q \vee \neg p,\ \ C_5 = u \vee \neg r,\ \ C_6 = \neg u,\ \ C_7 = \neg t\}$$

$$S = \{C_1,\ C_2,\ C_3,\ C_4,\ C_5\}$$

directed	
$R_1 = s$	(C_1, C_7)
$R_2 = p$	(R_1, C_2)
$R_3 = q$	(R_2, C_4)
$R_4 = r$	(R_3, C_3)
$R_5 = u$	(R_4, C_5)
$R_6 = \square$	(R_5, C_6)

non-directed	
$R_1 = t \vee p$	(C_1, C_2)
$R_2 = p$	(R_1, C_7)
$R_3 = q$	(R_2, C_4)
$R_4 = r$	(R_3, C_3)
$R_5 = u$	(R_4, C_5)
$R_6 = \square$	(R_5, C_6)

Directed Resolution (Wos-Robinson-Carson, 1965)

Properties

Directed resolution is *complete*: if $UNSAT(\mathcal{C})$ and $S \subset \mathcal{C}$ is satisfiable, then there exists a directed refutation of $\mathcal{C}$ with support set S

- this is not so useful if no way to find a satisfiable S is given

Heuristic for finding S

In practice, when trying a refutation of a conclusion from a set of premises, it is reasonable to consider the premises satisfiable

- premises: S
- negation of the conclusion (clause form): $\mathcal{C} \setminus S$
- if the premises are inconsistent, then every result can be derived
- yet, otherwise, $\Box$ can be derived from negating the conclusion

Ordered Resolution

Ordered derivations

An ordered derivation of C_m from $\{C_1, .., C_n\}$ is a sequence

$$\langle C_1, .., C_n, C_{n+1}, .., C_m \rangle$$

such that

- for every $i > n$, C_i is the resolvent of two previous clauses

  $$A_1 \vee L_{11} \vee .. \vee L_{1p} \qquad \text{and} \qquad \neg A_2 \vee L_{21} \vee .. \vee L_{2q}$$

  where A_1 and A_2 are unifiable with $MGU\ \sigma$ and order matters
- the literals of C_i are ordered as:

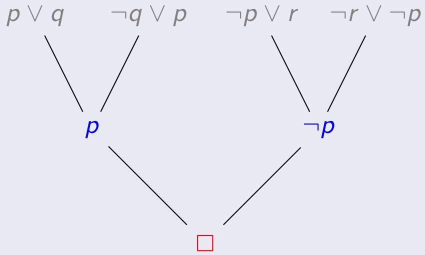
  $$(L_{11} \vee .. \vee L_{1p} \vee L_{21} \vee .. \vee L_{2q})\sigma$$

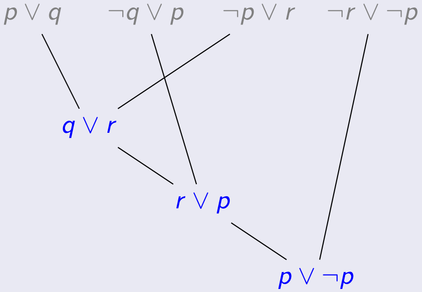
Ordered Resolution

(Non-)Properties

Ordered resolution is *not* complete

$$\text{counterexample:} \quad \{p \vee q, \;\; \neg q \vee p, \;\; \neg p \vee r, \;\; \neg r \vee \neg p\}$$





Summary

Correctness and completeness

- Correctness: $\square$ can be derived only if $UNSAT(\mathcal{C})$
- Completeness: if $UNSAT(\mathcal{C})$, then $\square$ can be derived

	correct	complete
linear	✓	✓
input	✓	no
directed	✓	✓(if $SAT(S)$)
ordered	✓	no