Exercises of Computational Logic

Notes

– It is possible that, by mistake, some exercises (or parts of them) are written in Spanish; in this case, hopefully, it should be not too difficult to understand and solve them!
– There can be mistakes of any type in the text of the exercises, especially in those which are translated from Spanish; please be patient.
– It is even possible that you will find the same exercise (or parts of it) twice or more.
– Basically, every time you have to prove the (un)satisfiability of a set of clauses, you can repeat the exercise using any other method you know; therefore, most exercises can be fruitfully done more than once using different techniques.
– You can compute the clause form of any formula written here, not only of those written in the corresponding section.

1 First-Order Logic

Exercise 1 *Consider the following alphabet:*

- *variable symbols:* x, y, z
- *function symbols:* $a/0$, $b/0$, $c/0$, $d/0$, $e/0$
- *predicate symbols:* $p_f/1$, $p_m/1$, $q_m/2$, $=/2$, $p_r/3$

Formalize the following sentences by means of first-order logic:

1. a *is* p_m
2. c *is* p_f *or* d *is* q_m *with* b
3. c *is* p_f *or* d *is* q_m *with someone between* b *and* e
4. a, b, c *are related by* p_r *in at least one way where* a *appears before* c *in the triple*
5. *if* b *is* p_f*, then someone between* a *and* b *is* p_f *and* p_m
6. c *is* q_m *with* e*, in both directions (i.e.,* c *is* q_m *with* e*, and* e *is* q_m *with* c*)*
7. a*,* b*, and* e *are all different*
8. *both* a *and* b *are* q_m *with someone, and* a *comes always first in the pair (i.e., that* a *is* q_m *with someone, not that someone is* q_m *with* a*)*
9. *all predicates with arity greater than one are commutative (Note: as it is, this sentence does not represent a first-order idea!)*
10. *someone is* p_f *but not* q_m *with anyone (in any direction)*
11. *any two equal individuals are* p_r *with at least another one (which will be the third element of triple) which is not equal to them*
12. *there is only one individual which is both* p_f *and* p_m

Exercise 2 *Translate the following formulæ into words (there are many possibilities!):*

1. $q_m(a, d)$
2. $p_f(a)$
3. $p_r(a, b, c) \vee p_r(b, a, c)$
4. $p_f(a) \vee p_f(b)$
5. $q_m(a, b) \rightarrow p_m(a) \wedge p_m(b)$
6. $\forall xy(q_m(x, y) \rightarrow p_m(x) \wedge p_m(y))$
7. $q_m(x, y) \rightarrow p_m(x) \wedge p_m(y)$
8. $\forall x(a = x)$
9. $\exists x p_f(x)$
10. $\exists x p_f(x) \vee \forall y(\neg p_f(y))$
11. $\forall x(p_f(x) \vee p_m(a))$
12. $\neg\exists y(\forall x p_r(x, c, y))$
13. $\exists xy(q_m(x, x) \wedge q_m(y, y) \wedge (\forall z(q_m(z, z) \rightarrow z = x \vee z = y)))$
14. $\forall x \forall x' \forall x''(= (x, x') \wedge = (x', x'') \rightarrow x'' = x)$

Exercise 3 *Consider the domain*

$$D = \{\textit{Jim Henle}, \textit{Tom Tymoczko}, \textit{Aristotle}, \textit{Adrienne Rich}, \textit{Madonna}\}$$

and the following facts:

- a *refers to Jim Henle (a mathematical logician)*
- b *refers to Madonna (a singer* $\wedge$ *a dancer but* $\neg$ *a great actress)*
- c *refers to Tom Tymoczko (a philosophical logician)*
- d *refers to Aristotle (a philosopher* $\wedge$ *a scientist* $\wedge$ *a logician)*
- e *refers to Adrienne Rich (a poet)*
- $p_f(x)$ *means that* x *is a female*
- $p_m(x)$ *means that* x *is a male*
- $q_m(x, y)$ *means that* x *is married to* y
- $p_r(x, y, z)$ *means that* x *and* y *are parents of* z

Define new predicates to formalize the properties about these individuals (that someone is a dancer, a poet, etc.).

Exercise 4 *Using also the predicates defined in Ex. 3, formalize that*

1. *Madonna is married to Aristotle*
2. *Aristotle is male*
3. *Aristotle is married to Madonna*
4. *Jim Henle is Madonna*
5. *Madonna is herself*
6. *everyone is herself (himself)*
7. *x is a child of Aristotle and Tom Tymoczko*
8. *Adrienne Rich is not herself*
9. *Jim Henle is male and Tom Tymoczko is female*
10. *Tom Tymoczko is not married to Jim Henle*
11. *Tom Tymoczko is not married to anyone*
12. *there is a logician, which is a philosopher but is not Aristotle, which is married to a poet*
13. *if Madonna were a philosopher, then she would be married to a logician*
14. *all scientists are logicians*
15. *all scientists are logicians, apart from men*

Only after building the formulæ, say whether they are true or false in the interpretation above (and assuming the knowledge of the real world)

Exercise 5 *For each of the furmulæ you built in Exercise 4, find, when possible, another domain and another interpretation of constants and predicates (as simple as possible) such that the truth value of the formula changes (from* **t** *to* **f***, and from* **f** *to* **t***)*

Exercise 6

1. *does* 5 *belong to the following set?*

$$X = \{n' \in \mathbf{N} \setminus \{n'' \mid n'' \leq 4\} \mid n' \text{ is greater than 7 whenever it is even}\}$$

2. *what about* 6*?*
3. *and* 8*?*
4. *how would you write* $p_X(m) =$ *"m belongs to X" as a logical formula?*
5. *why don't we specify from which set we pick* n'' *in* $\{n'' \mid n'' \leq 4\}$*?*

Exercise 7 *Check the correctness of this logical deduction by considering all possible interpretations:*

$$\{p \vee (q \wedge r),\ p \rightarrow \neg q,\ \neg p \rightarrow \neg r\} \models p$$

What about the domain of the interpretations? why don't we need to mention it?

Exercise 8 (04/2009) *Let A, B, C, D, E, F, G formulæof a first-order language, about which the following is known (note that nothing is known about E):*

A is a tautology
D is the negation of C
B is unsatisfiable
F is false for a given interpretation I
there are model and countermodels of C
G is true for the same I

For each of the following statements, say if it is correct (YES), incorrect (NO), or it is not possible to find an answer (UNK).

1	$D \wedge C$ *is true in the given interpretation I*
2	$A \wedge E$ *is satisfiable but not valid*
3	*there are no models of* $A \wedge C$
4	$D \rightarrow C$ *is true in the given interpretation I*
5	$A \wedge ((G \wedge \neg C) \rightarrow (D \wedge G))$ *is true in the given interpretation I*
6	$A \wedge (B \vee G)$ *is valid*
7	$G \vee B$ *has models*
8	$F \vee (A \wedge C)$ *is true in the given interpretation I but is not valid*
9	*the negation of* $A \wedge C \rightarrow B$ *is valid*
10	$F \rightarrow (G \vee E)$ *is satisfiable*

Solution 1

1 *NO ($D = \neg C$ implies that C and D do not share models)*
2 *UNK (no information about E)*
3 *NO (there exist models of C, which are also models of A)*
4 *UNK (nothing is known about C and D in I)*
5 *YES (A has only models, and the implication is true in both cases (C true and D false, and the other way around) since G is true in I*
6 *UNK (we know that G is satisfiable but we don't know if it is valid (it must be in order for $A \wedge (B \vee G)$ to be valid)*
7 *YES (I is a model)*
8 *UNK (the formula is satisfiable, but we don't know if I is a model)*
9 *NO (it is the same as saying that $A \wedge C \rightarrow B$ is unsatisfiable, but there are countermodels of C, so that there is a mo*
10 *YES (G has models (I))*

Exercise 9 (02/2008) *Let A, B, C, D, E, F, G be formulæ defined within the same first-order language. The following is all we know about them*

- *A is a tautology*
- *B is a contradiction*
- *C is satisfiable*
- *D is the negation of C*
- *we know nothing about E*
- *F is false for a particular interpretation I*
- *G is false for a particular interpretation I*

For each of the following statements, answer Y (for yes), N (for no) or U (for unknown, impossible to say with the given information):

1. *$D \wedge C$ is true for the particular interpretation I*
2. *$F \rightarrow G$ is satisfiable*
3. *$A \wedge C$ is satisfiable*
4. *$A \wedge C$ is true for the particular interpretation I*
5. *$D \rightarrow C$ is true for the particular interpretation I*
6. *$D \vee C$ is true for the particular interpretation I*
7. *$A \wedge B$ is satisfiable*
8. *$A \wedge E$ is satisfiable*
9. *$G \wedge F$ is satisfiable*
10. *$A \wedge C \rightarrow B$ is a contradiction*

2 Standardization of formulæ

Exercise 10 (09/1993) *Consider the set of formulæ* $\{A_1, A_2, A_3, A_4\}$*, being:*

$$\begin{array}{l} A_1 : \forall z \exists x((\neg P(z) \vee \exists y Q(x, y, z)) \wedge (\neg R(z) \vee \neg P(x))) \\ A_2 : \exists z S(z) \rightarrow \forall z T(z) \\ A_3 : \forall z \forall x(T(z) \rightarrow P(x)) \\ A_4 : \exists z \forall y(\neg \exists x Q(z, x, y) \wedge S(y) \wedge R(z)) \end{array}$$

where x*,* y*,* z *are variable symbols: write them in clause form.*

Solution 2 *See Luís Iraola's resolved exercises.*

Exercise 11 (09/1994) *Obtain the clause form of each of the following:*

$$\begin{array}{l} A_1 : \exists x(\exists y A(x, y) \rightarrow \neg \forall z(B(x, z) \wedge C(z))) \\ A_2 : \neg(\exists x A(x) \wedge \neg \exists y \forall x C(x, y)) \end{array}$$

Exercise 12 (02/1994) *Obtain the clause form of the formula:*

$$\forall x(\exists y P(x, y) \vee \neg \forall y(Q(y) \rightarrow \exists x R(y, x)))$$

Exercise 13 (09/1997) *Given the formulæ:*

$$\begin{array}{l} P_1 : \forall x \forall y(M(x, y) \rightarrow Z(x, y)) \\ P_2 : \forall x \forall y(V(x) \rightarrow M(x, y)) \\ P_3 : \forall x \forall y \forall z(Z(x, y) \wedge Z(y, z) \rightarrow Z(x, z)) \\ P_4 : \forall x \forall y(L(x) \wedge B(y) \rightarrow Z(x, y)) \\ P_5 : \forall x \forall y(B(x) \wedge P(y) \rightarrow Z(x, y)) \\ P_6 : \forall x(P(x) \rightarrow V(x)) \end{array}$$

1. *Build the clause set corresponding to the formula* $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge P_6$*.*
2. *Give the clause set which is needed in order to study the correctness of* $T[P_1, P_2, P_3, P_4, P_5, P_6] \vdash Q$*, being*

$$Q : \forall x \forall y(\exists z B(z) \wedge L(x) \wedge P(y) \rightarrow Z(x, y))$$

Exercise 14 (09/1998) *Let* $\{A_1, A_2, A_3, A_4, A_5\}$ *be the following set of formulæ:*

$$\begin{array}{l} A_1 : \forall x \exists y(A(x, y) \rightarrow B(x) \wedge C(y)) \\ A_2 : \neg \forall y C(y) \\ A_3 : \forall x(B(x) \rightarrow \exists x \exists y \neg D(x, y)) \\ A_4 : \forall x \neg E(x) \\ A_5 : \forall x \forall y D(x, y) \vee \exists x E(x) \end{array}$$

Build the corresponding clause set.

Solution 3 *See Luís Iraola's resolved exercises.*

Exercise 15 (09/1998) *Let* $\{A_1, A_2, A_3, A_4\}$ *be the following set of formulæ:*

$$\begin{array}{l} A_1 : \exists x B(x) \\ A_2 : \forall x \forall y(C(x, y) \rightarrow D(x) \wedge B(x)) \\ A_3 : \exists x \forall y C(x, y) \\ A_4 : \neg \exists x \exists y(D(x) \wedge \neg A(y)) \end{array}$$

Build the corresponding clause set.

Exercise 16 (06/1998) *Let* $\{A_1, A_2, A_3, A_4\}$ *be the following set of formulæ:*

$$\begin{array}{l} A_1 : R(b) \wedge G(b) \wedge R(a) \\ A_2 : \forall x(M(x) \rightarrow (G(x) \rightarrow P(x)) \wedge (\neg G(x) \rightarrow \neg P(x))) \\ A_3 : \forall x(R(x) \rightarrow M(x)) \\ A_4 : \neg \exists x(R(x) \wedge \neg D(x)) \end{array}$$

Build the corresponding clause set.

Exercise 17 (02/1999) *Let $\{A_1, A_2, A_3\}$ the set of formulæ corresponding to a statement.*

$$A_1 : \exists y \forall x (A(x,y) \to B(x) \land D(x,y))$$
$$A_2 : \forall x (B(x) \to \exists y \neg D(y,x) \lor E(x))$$
$$A_3 : \forall x \neg E(x)$$

Build the corresponding set of clauses.

Exercise 18 (06/1999) *Build the set of clauses corresponding to the set $\{A_1, A_2, A_3\}$:*

$$A_1 : \exists x \forall y (A(x) \to B(x) \land (C(y) \land D(x,y)))$$
$$A_2 : \forall x \exists y (A(y) \land D(x,y))$$
$$A_3 : \neg \exists x \forall y (\neg C(x) \lor (B(a) \land D(x,y))) \land \neg \exists x A(x)$$

Exercise 19 (09/1999) *Given the following formulæ:*

$$F_1 : \forall x (\forall y (A(x,y) \lor B(y)) \to C(x) \lor D(x))$$
$$F_2 : \forall x \exists y (C(y) \to A(y,x))$$
$$F_3 : \forall x (B(x) \lor D(x))$$
$$F_4 : \forall x \forall y \neg (\neg A(x,y) \to D(x) \lor D(y))$$

Put $F_1 \land F_2 \land F_3 \land F_4$ in clause form.

Solution 4 *See Luís Iraola's resolved exercises.*

Exercise 20 (09/2000) *Consider a theory whose non-logical axioms are as follows:*

$$F_1 : \forall x (A(x) \to D(x) \land E(x))$$
$$F_2 : \neg \forall x \exists y (A(x) \land B(x) \to \neg A(y))$$
$$F_3 : \forall x (D(x) \to (B(x) \leftrightarrow C(x)))$$

Compute the clause form of such theory.

Exercise 21 (06/2000) *Compute the clause form of the deductive structure: $[P_1, P_2] \vdash C$*

$$P_1 : \forall x \exists y \exists z (P(x,z) \lor Q(x) \to (P(y) \to \forall x R(x)))$$
$$P_2 : \forall x \exists y (R(x) \land \neg Q(y)) \to \exists x \forall y (\neg P(x,y) \lor P(y,x))$$
$$C : \forall x P(x) \land \forall y Q(y)$$

Solution 5 *See Luís Iraola's resolved exercises.*

Exercise 22 (09/2001) *Compute the clause form of the following deductive structure: $[C_1, C_2] \vdash Q$*

$$C_1 : \exists x \neg (A(x) \to \exists y (\neg C(y) \to \neg B(y,x))) \land \forall x (\neg D(x) \to \neg C(x))$$
$$C_2 : \forall x (A(x) \land \neg E(x) \to \exists y (B(y,x) \land \neg D(y)))$$
$$Q : \forall x \neg (\exists y (B(y,x) \land C(y)) \land \neg E(x) \land A(x))$$

Solution 6 *See Luís Iraola's resolved exercises.*

Exercise 23 (09/2002) *Given the following set of formulæ:*

$$F_1 : \forall x (q(x) \to \exists y \neg r(x,y) \lor s(x))$$
$$F_2 : \exists y \forall x (p(x,y) \to q(x) \land r(y,x))$$
$$F_3 : \forall x \forall y p(x,y)$$
$$F_4 : \forall x \neg s(x)$$

build the corresponding set of clauses.

Exercise 24 (06/2003) *Rewrite the following clauses, so that they contain at least one implication, and they contain neither constants nor terms with functions:*

$$C_1 : R(y) \lor \neg Q(f(x), x)$$
$$C_2 : \neg B(x) \lor R(y) \lor R(g(x,y))$$
$$C_3 : B(x) \lor R(b) \lor Q(x) \lor H(b)$$
$$C_4 : R(y) \lor \neg H(b)$$

Exercise 25 (02/2003) *Given the following set of formulæ:*

$$F_1 : \forall x(A(x) \land \exists y \neg B(y) \to C(x, y))$$
$$F_2 : \exists x A(x) \land \neg \forall y \exists z C(z, y)$$
$$F_3 : \forall y \exists x \forall z((B(x) \to A(z)) \land (\neg C(y, z) \to \neg B(y)))$$

Build the corresponding clause set.

Solution 7 *See Luís Iraola's resolved exercises.*

Exercise 26 (09/2003) *For each of the following formulæ, mark with X the answers corresponding to CORRECT clause forms of the initial formula.*

$$\forall x(A(x) \land B(f(x)) \to \forall z C(z, x))$$

	CORRECT	*INCORRECT*
$\neg A(x) \lor \neg B(f(x)) \lor C(z, x)$	☐	☐
$\neg A(a) \lor \neg B(f(a)) \lor C(z, a)$	☐	☐
$\neg A(g(z)) \lor \neg B(f(g(z))) \lor C(z, g(z))$	☐	☐
$\neg A(a) \lor \neg B(f(a)) \lor C(z, b)$	☐	☐
$\neg A(g(z)) \lor \neg B(f(g(z))) \lor C(z, a)$	☐	☐

$$\exists x A(x) \land \exists x B(x) \land \forall y C(y)$$

	CORRECT	*INCORRECT*
$A(a), B(a), C(y)$	☐	☐
$A(a), B(b), C(f(a))$	☐	☐
$A(f(y)), B(f(y)), C(y)$	☐	☐
$A(f(y)), B(a), C(y)$	☐	☐
$A(a), B(b), C(y)$	☐	☐

$$\neg(\forall x \exists y A(x, y) \to \exists z B(z)) \land \neg(\exists z B(z) \to \forall y \exists x C(x, f(y)))$$

	CORRECT	*INCORRECT*
$A(x, g(x)), \neg B(z), B(a), \neg C(b, f(y))$	☐	☐
$A(x, g(x)), \neg B(a), B(a), \neg C(b, f(y))$	☐	☐
$A(x, g(x)), \neg B(z), B(a), \neg C(x, f(b))$	☐	☐
$A(x, g(x)), B(a), \neg C(x, f(b))$	☐	☐
$A(g(x), x), \neg B(a), \neg C(x, f(b))$	☐	☐

Exercise 27 (06/2004) *Put in clause form the following deductive structure:* $[P_1, P_2] \vdash C$

$$P_1 : \exists x A(x) \lor \exists x B(x) \to \exists x C(x)$$
$$P_2 : \forall x A(x) \to (\exists y B(y) \to (\forall z C(z) \to A(a)))$$
$$C : \forall y(\exists x A(x) \to (B(y) \land A(y) \to \exists z B(z)))$$

Exercise 28 (06/2006) *For each of the following formulæ, mark with X the answers corresponding to CORRECT clause forms of the initial formula.*

$$(\forall x \exists y \forall z(A(x) \lor \neg B(y, z)) \to \exists x \forall t B(x, t)) \land \neg(B(a) \lor \forall s A(s))$$

	CORR.	*INCORR.*
$\neg A(b) \lor B(b, t), B(y, f(y)) \lor B(b, t), \neg B(a), \neg A(c)$	☐	☐
$\neg A(b) \lor B(c, t), B(y, f(y)) \lor B(c, t), \neg B(a), \neg A(g(y))$	☐	☐
$\neg A(b) \lor B(f(y), t), B(y, g(y)) \lor B(f(y), t), \neg B(a), \neg A(c)$	☐	☐
$\neg A(f(t)) \lor B(b, t), B(y, g(y, t)) \lor B(b, t), \neg B(a), \neg A(c)$	☐	☐
$\neg A(c) \lor B(b, t), B(y, f(t)) \lor B(b, t), \neg B(a), \neg A(a)$	☐	☐

$$\exists x A(x) \lor \exists x B(x) \lor \exists x \forall y C(x, y, f(a))$$

	CORR.	*INCORR.*
$A(a), B(b), C(c, y, f(a))$	☐	☐
$A(b), B(b), C(b, y, f(a))$	☐	☐
$A(b), B(b), C(c, y, f(a))$	☐	☐
$A(g(y)), B(g(y)), C(b, y, f(a))$	☐	☐
$A(c), B(g(y)), C(b, y, f(a))$	☐	☐

$$\forall y(A(y) \to \exists z B(z,a)) \wedge \forall t A(t) \to \exists x B(a,x)$$

	CORR.	*INCORR.*
$A(b) \vee \neg A(b) \vee B(a,c), \neg B(z,a) \vee \neg A(b) \vee B(a,c)$	☐	☐
$A(y) \vee \neg A(t) \vee B(a,f(y,t)), \neg B(g(y,t),a) \vee \neg A(t) \vee B(a,f(y,t))$	☐	☐
$A(b) \vee \neg A(a) \vee B(a,f(z)), \neg B(z,a) \vee \neg A(a) \vee B(a,f(z))$	☐	☐
$A(b) \vee \neg A(f(z)) \vee B(a,c), \neg B(z,a) \vee \neg A(f(z)) \vee B(a,c)$	☐	☐
$A(y) \vee \neg A(b) \vee B(a,c), \neg B(f(y),a) \vee \neg A(t) \vee B(a,x)$	☐	☐

Exercise 29 (06/2007) *For each of the following formulæ, mark with X the answers corresponding to CORRECT clause forms of the initial formula.*

$$\forall x(\forall y(\forall z A(z) \to \neg \forall t B(y,t)) \to \exists y B(x,y))$$

	CORR.	*INCORR.*
$A(z) \vee B(x,f(x)), B(f(x),t) \vee B(x,f(x))$	☐	☐
$A(z) \vee B(x,f(x)), B(y,a) \vee B(x,f(x))$	☐	☐
$A(z) \vee B(x,f(x)), B(g(x),t) \vee B(x,f(x))$	☐	☐
$A(z) \vee B(x,f(x,z,t)), B(g(x),t) \vee B(x,f(x,z,t))$	☐	☐

$$\forall x \exists y A(x,y) \vee \forall x \exists z B(x,z)$$

	CORRECT	*INCORRECT*
$A(x,f(x)) \vee B(s,z)$	☐	☐
$A(x,f(x,s)) \vee B(s,f(x,s))$	☐	☐
$A(x,f(x)) \vee B(x,g(x))$	☐	☐
$A(x,f(x)) \vee B(x,a)$	☐	☐

Exercise 30 (02/2008) *Compute the clause form of the following formulæ:*

$$A_1 : \forall x(P(x) \to Q(x) \vee Q(a)) \to (\exists x P(x) \to \exists x(Q(x) \vee Q(a)))$$
$$A_2 : \neg \exists x \forall z(P(x) \to \neg Q(z)) \vee (\exists z A(y,z) \to \exists u B(y,u))$$
$$A_3 : \neg \exists x \forall y(\neg C(x) \vee (B(a) \wedge D(x,y))) \wedge \neg \exists x A(x)$$

Exercise 31 *Prove that the following equivalence rules are indeed valid formulæ:*

$$(\forall x F \to G) \leftrightarrow \exists x(F \to G)$$
$$(\exists x F \to G) \leftrightarrow \forall x(F \to G)$$
$$(F \to \forall x G) \leftrightarrow \forall x(F \to G)$$
$$(F \to \exists x G) \leftrightarrow \exists x(F \to G)$$

Hint: prove each direction by supposing it not to hold and deriving a contradiction.

Exercise 32 *Find a proof for the Lemma on the existence of the prenex form: The prenex form of a formula always exists.*
Hint: consider that finding a prenex form consists of applying a set of equivalence rules, in a certain direction...

Exercise 33 *Is there any case where the prenex form is not unique, in a non-trivial way? that is, can two prenex forms of the same formulæ be different apart from the names of bounded variables?*

Also think about whether the prenex form of F may be defined as

– *a formula which is in prenex form and is equivalent to F; or*
– *a formula which is in prenex form and is obtained from F by applying a certain set of equivalence rules.*

Exercise 34 *Find a proof for the Lemma on the existence of the conjunctive normal form: The conjunctive normal form of a quantifier-free formula always exists.*
Hint: consider that finding a CNF consists of applying a set of equivalence rules, in a certain direction...

Exercise 35 *Consider the formula*

$$\neg((F_1 \wedge G_1) \vee (F_2 \wedge G_2))$$

By applying the rules for getting a conjunctive normal form, it is possible to get

$$(\neg F_1 \vee \neg G_1) \wedge (\neg F_2 \vee \neg G_2)$$

but also a much more complicated one (guess how?), which happens to be more restrictive than the first one: it takes the form

$$... \wedge (\neg F_1 \vee \neg G_1) \wedge (\neg F_2 \vee \neg G_2) \wedge ...$$

- *why is it more restrictive (look at the structure)?*
- *how is it possible? shouldn't they be equivalent (they are both equivalent to the original formula)? where is the trick?*

Exercise 36 *Compute the Skolem normal form of the following formulæ:*

1. $\exists x \forall y \forall z \exists u \forall v \exists w p(x, y, z, u, v, w)$
2. $\forall x \exists y \exists z((\neg p(x, y) \wedge q(x, z)) \vee r(x, y, z))$
3. $\neg(\forall x p(x) \rightarrow \exists y \forall z q(y, z))$
4. $\forall x((\neg e(x, 0) \rightarrow (\exists y(e(y, g(x)) \wedge \forall z(e(z, g(x)) \rightarrow e(y, z))))))$
5. $\neg(\forall x p(x) \rightarrow \exists y p(y))$

Exercise 37 (02/2009) *Obtain the clause form of the following two formulæ:*

1. $p(z) \rightarrow \neg q(z) \vee (\forall x(\forall y p(y) \rightarrow r(x, y)))$
2. $p(x) \wedge \forall z\,((\exists y q(z, y) \rightarrow q(f(f(y)), f(z))) \rightarrow r(z, f(z), x))$

and, for each of them,

- *highlight the trasformation steps from the original formula to its clause form;*
- *specify which property is preserved through each step, and which property is preserved throughout the entire trasformation.*

Solution 8 *Note that, in both formulæ, there is a quantifier (the $\forall y$ in $\forall y p(y)$, and the $\exists y$ in $\exists y q(z, y)$ in the second) which has a smaller scope than it looks like at a first glance. Consequently, some occurrences of y are free (the one in $r(x, y)$ in the first formula, and the one in $q(f(f(y)), f(z))$ in the second). We choose to rename the bounded y as y'. Note also that, strictly speaking, renaming the free occurrences of y (instead of the bounded ones) is not allowed since the meaning of free variables is given from outside.*

1. $p(z) \rightarrow \neg q(z) \vee (\forall x(\forall y' p(y') \rightarrow r(x, y)))$
   - *prenex form:*
     $\forall x \exists y'(p(z) \rightarrow \neg q(z) \vee (p(y') \rightarrow r(x, y)))$
   - *existential closure:*
     $\exists z \exists y \forall x \exists y'(p(z) \rightarrow \neg q(z) \vee (p(y') \rightarrow r(x, y)))$
   - *CNF:*
     $\exists z \exists y \forall x \exists y'(\neg p(z) \vee \neg q(z) \vee \neg p(y') \vee r(x, y))$
   - *Skolem:*
     $\forall x(\neg p(a) \vee \neg q(a) \vee \neg p(f(x)) \vee r(x, b))$
   - *clause form:*
     $\{\ \neg p(a) \vee \neg q(a) \vee \neg p(f(x)) \vee r(x, b)\ \}$
2. $p(x) \wedge \forall z\,((\exists y' q(z, y') \rightarrow q(f(f(y)), f(z))) \rightarrow r(z, f(z), x))$
   - *prenex form:*
     $\forall z \exists y'\,(p(x) \wedge ((q(z, y') \rightarrow q(f(f(y)), f(z))) \rightarrow r(z, f(z), x)))$
   - *existential closure:*
     $\exists x \exists y \forall z \exists y'\,(p(x) \wedge ((q(z, y') \rightarrow q(f(f(y)), f(z))) \rightarrow r(z, f(z), x)))$
   - *CNF:*
     $\exists x \exists y \forall z \exists y'\,(p(x) \wedge (q(z, y') \vee r(z, f(z), x)) \wedge (\neg q(f(f(y)), f(z)) \vee r(z, f(z), x)))$
   - *Skolem:*
     $\forall z\,(p(a) \wedge (q(z, g(z)) \vee r(z, f(z), a)) \wedge (\neg q(f(f(b)), f(z)) \vee r(z, f(z), a)))$
   - *clause form:*
     $\{\ p(a),\ q(z, g(z)) \vee r(z, f(z), a),\ \neg q(f(f(b)), f(z)) \vee r(z, f(z), a)\ \}$

Satisfiability is preserved everywhere. Semantics is preserved in obtaining the prenex form and the CNF, not necessarily in the existential closure and in the introduction of Skolem functions.

Exercise 38 (03/2009) *Given the following formula and each of its possible clause forms below, mark with an X the correct answer, depending on whether the clause form is a correct (C) or an incorrect (I) one for the formula:*

$$\exists z(\forall x p(x,z) \land \exists z r(g(h(a)),z) \to q(y,z,b)) \to r(z,c)$$

		C	I
(1)	$r(b,c) \lor p(l(w,v),w),\ \neg q(y,w,b) \lor r(b,c),\ r(g(h(a)),v) \lor r(b,c)$	□	□
(2)	$r(d,c) \lor p(f(z'),z'),\ \neg q(y,z',b) \lor r(d,c),\ r(g(h(a)),z'') \lor r(d,c)$	□	□
(3)	$p(x,w) \lor r(f,c),\ \neg q(y,v,b) \lor r(f,c),\ r(g(h(a)),v) \lor r(f,c)$	□	□
(4)	$\neg p(x,z') \lor r(d,c) \lor r(g(h(a)),z''),\ \neg p(x,z') \lor r(d,c) \lor \neg q(y,z',b)$	□	□
(5)	$r(e,c) \lor p(f(v,w),w),\ r(e,c) \lor \neg q(y,w,b),\ r(e,c) \lor r(g(h(a)),v)$	□	□

Solution 9

(1): incorrect since the names for the Skolem functions are not fresh
(2): correct
(3): incorrect since the scope of the second $\exists z$ does not include $q(y,z,b)$
(4): incorrect because $\to$ has a lower precedence than $\land$
(5): correct

Exercise 39 (03/2009) *Obtain the clause form of the following deductive structure: $[P_1, P_2] \vdash C$*

P_1 : $\forall w\,((q(w) \land (\exists y p(w,g(y)) \to \exists x p(g(x),w))) \lor \forall z r(z,w))$
P_2 : $\exists v\,(q(v) \to \neg\exists z p(g(z),z) \land \neg\forall v(\neg r(v,v)))$
C : $\exists x r(g(x),g(g(b))) \land \forall v p(a,v)$

Solution 10

P_1^1 $\forall w \forall y \exists x \forall z\,((q(w) \land (p(w,g(y)) \to p(g(x),w))) \lor r(z,w))$
P_1^2 *the same*
P_1^3 $\forall w \forall y \exists x \forall z\,((q(w) \lor r(z,w)) \land (\neg p(w,g(y)) \lor p(g(x),w) \lor r(z,w)))$
P_1^4 $\forall w \forall y \forall z\,((q(w) \lor r(z,w)) \land (\neg p(w,g(y)) \lor p(g(h(w,y)),w) \lor r(z,w)))$
P_1^C $q(w) \lor r(z,w),\quad \neg p(w,g(y)) \lor p(g(h(w,y)),w) \lor r(z,w)$

P_2^1 $\exists v \forall z \exists v'(q(v) \to \neg p(g(z),z) \land r(v',v'))$
P_2^2 *the same*
P_2^3 $\exists v \forall z \exists v'((\neg q(v) \lor \neg p(g(z),z)) \land (\neg q(v) \lor r(v',v')))$
P_2^4 $\forall z((\neg q(c) \lor \neg p(g(z),z)) \land (\neg q(c) \lor r(f(z),f(z))))$
P_2^C $\{\neg q(c) \lor \neg p(g(z),z),\quad \neg q(c) \lor r(f(z),f(z))\}$

C^1 $\forall x \exists v(\neg r(g(x),g(g(b))) \lor \neg p(a,v))$
C^2 *the same*
C^3 *the same*
C^4 $\forall x(\neg r(g(x),g(g(b))) \lor \neg p(a,l(x)))$
C^C $\{\neg r(g(x),g(g(b))) \lor \neg p(a,l(x))\}$

Exercise 40 (03/2009) *For each of the following formulæ and each of their possible clause form which are written, tell if they are correct or incorrect clause forms:*

– $\forall x((B(x) \to \exists y \neg D(y,x) \lor C(x)) \to \forall z C(z)) \to \forall x A(x,y)$

$\neg B(a) \lor \neg D(b,a) \lor C(a) \lor A(x,d), \neg C(c) \lor A(x,d)$
$B(a) \lor \neg C(z) \lor A(x,b), D(y,a) \lor \neg C(z) \lor A(x,b), \neg C(a) \lor \neg C(z) \lor A(x,b)$
$\neg B(f(x)) \lor \neg D(g(x),f(x)) \lor C(f(x)) \lor A(x,a), \neg C(h(x)) \lor A(x,a)$
$\neg B(a) \lor \neg D(y,a) \lor C(a) \lor A(a,b), \neg C(f(y)) \lor A(a,b)$
$\neg B(a) \lor \neg D(b,a) \lor C(a) \lor A(x,c), \neg C(f(x)) \lor A(x,c)$

– $\forall x(A(x) \wedge \exists y B(f(x), y)) \rightarrow \exists z C(z, x)$

$$\begin{array}{l} \neg A(a) \vee \neg B(f(a), y) \vee C(z, a) \\ \neg A(a) \vee \neg B(f(a), y) \vee C(c, b) \\ \neg A(x) \vee \neg B(f(x), y) \vee C(f(y), x) \\ \neg A(a) \vee \neg B(f(a), y) \vee C(g(y), b) \\ \neg A(a) \vee \neg B(f(a), y) \vee C(g(y), a) \end{array}$$

Exercise 41 (03/2009) *Obtain the clause form of the following deductive structure* $[P_1, P_2] \vdash C$*:*

$$\begin{array}{r} P_1 : \exists x(A(x) \vee \forall y B(x, y)) \rightarrow \neg \forall z(A(z) \rightarrow C(x, z)) \\ P_2 : \forall x \exists y(A(x) \wedge \neg A(y)) \rightarrow \exists x \forall y(\neg B(x, y) \vee C(y, x)) \\ C : \forall x(A(x) \rightarrow \exists x \exists y \neg C(x, y)) \end{array}$$

Exercise 42 (03/2009) *For the following formula and each of the possible clause forms below, say whether the clause form is a correct one for the formula:*

$$\forall x((\neg A(x, a) \rightarrow (\exists y(B(y, g(x), z) \wedge \forall z(B(z, g(x)) \rightarrow A(y, z))))))$$

$$\begin{array}{l} A(x, a) \vee B(f(x, z), g(x), z), A(x, a) \vee \neg B(z, g(x)) \vee A(f(x, z), z) \\ A(x, a) \vee B(f(x), g(x), z), \neg B(z, g(x)) \vee A(f(x), z) \\ A(x, a) \vee B(f(x), g(x), b), A(x, a) \vee \neg B(z, g(x)) \vee A(f(x), z) \\ A(x, a) \vee B(f(z), g(x), z), A(x, a) \vee \neg B(z, g(x)) \vee A(f(z), z) \\ A(b, a) \vee B(c, g(x), z), A(b, a) \vee \neg B(z, g(x)) \vee A(c, z) \end{array}$$

Exercise 43 (03/2009) *Obtain the clause form of the following deductive structure* $[P_1, P_2] \vdash C$*:*

$$\begin{array}{r} P_1 : \neg \exists x \neg \exists y \neg (A(x) \rightarrow B(x, y)) \vee \exists x \forall z(B(x, z) \rightarrow C(y, z)) \\ P_2 : \neg \exists x \forall y(\neg A(x) \vee (D(a) \wedge B(x, y))) \wedge \neg \exists x D(x) \\ C : \forall x(B(x) \rightarrow \exists y \neg C(y, x) \vee D(x)) \end{array}$$

Exercise 44 (09/2008) *Obtain the clause form of the deductive structure* $[A_1, A_2] \vdash B$*, where:*

$$\begin{array}{r} A_1 : \forall x \exists y(A(x, y) \vee B(x, y)) \rightarrow \exists x \forall z C(x, z) \\ A_2 : \forall x D(x) \rightarrow \forall x \forall y A(x, y) \\ B : \exists x(\exists y(D(x) \wedge B(x, y)) \rightarrow \forall z C(x, z)) \end{array}$$

Exercise 45 (07/2009) *Obtain the clause form of each of the following formulæ(all independent):*

$$\begin{array}{l} (a)\ \forall x(\neg p(x, a) \rightarrow \exists y(p(y, g(x)) \wedge \forall z(p(z, g(x)) \rightarrow P(y, z)))) \\ (b)\ \forall x p(x) \wedge [(\forall y(q(y) \rightarrow \neg r(a, y)) \rightarrow \forall y p(y)) \vee \forall x p(x)] \end{array}$$

Solution 11

$$(a)\ \forall x(\neg p(x, a) \rightarrow \exists y(p(y, g(x)) \wedge \forall z(p(z, g(x)) \rightarrow P(y, z))))$$

$$\begin{array}{l} \forall x(p(x, a) \vee \exists y \forall z(p(y, g(x)) \wedge (\neg p(z, g(x)) \vee p(y, z)))) \\ \forall x \exists y \forall z[(p(x, a) \vee p(y, g(x))) \wedge (p(x, a) \vee \neg p(z, g(x)) \vee P(y, z))] \\ \textit{Clause form}: \{p(x, a) \vee p(f(x), g(x)), p(x, a) \vee \neg p(z, g(x)) \vee p(f(x), z)\} \end{array}$$

$$(b)\ \forall x p(x) \wedge [(\forall y(q(y) \rightarrow \neg r(a, y)) \rightarrow \forall y p(y)) \vee \forall x p(x)]$$

$$\begin{array}{l} \forall x p(x) \wedge [(\exists y(q(y) \wedge r(a, y)) \vee \forall y p(y)) \vee \forall x P(x)] \\ \forall x p(x) \wedge \exists y \forall z \forall t[((q(y) \vee p(z)) \wedge (r(a, y) \vee P(z))) \vee p(t)] \\ \forall x p(x) \wedge \exists y \forall z \forall t[(q(y) \vee p(z) \vee p(t)) \wedge (r(a, y) \vee p(z) \vee p(t))] \\ \forall x \exists y \forall z \forall t[p(x) \wedge (q(y) \vee p(z) \vee p(t)) \wedge (r(a, y) \vee p(z) \vee P(t))] \\ \textit{Clause form}: \{p(x), q(f(x)) \vee p(z) \vee p(t), r(a, f(x)) \vee P(z) \vee P(t)\} \end{array}$$

3 Standardization of interpretations

Exercise 46 (09/1993) *Given the set of formulæ* $\{F_1, F_2, F_3, F_4\}$

$$\begin{array}{l} F_1 : \forall z \exists x((\neg P(z) \vee \exists y Q(x,y,z)) \wedge (\neg R(z) \vee \neg P(x))) \\ F_2 : \exists z S(z) \to \forall z T(z) \\ F_3 : \forall z \forall x (T(z) \to P(x)) \\ F_4 : \exists z \forall y (\neg \exists x Q(z,x,y) \wedge S(y) \wedge R(z)) \end{array}$$

Starting from the clause form, and considering the Herbrand base and the Herbrand universe, build the associated semantic tree. What can be deduced about the satisfiability of the formulæ originating such tree?

Solution 12 *See Luís Iraola's resolved exercises.*

Exercise 47 (02/1994) *Starting from the clause form of the formula:*

$$\forall x(\exists y P(x,y) \vee \neg \forall y (Q(y) \to \exists x R(y,x)))$$

and considering tha definition of the Herbrand universe, build H_0*,* H_1 *and* H_2*.*

Exercise 48 *Study if the set of clauses associated with the following deductive structures is unsatisfiable. In case it is not, find a countermodel of the deductive structure, starting from the associated semantic tree.*

1. $p \wedge \neg r \to q, r \to s, q \vee s \to t \vdash \neg p$
2. $p \to \neg q, q \vee s, (p \to \neg r) \to s \vdash s$

(being p*,* q*,* r*,* s*,* t *closed or propositional formulæ).*

Solution 13 *See Luís Iraola's resolved exercises.*

Exercise 49 (06/1997) *If an Herbrand interpretation* I *is defined with the following truth values:*

$$\begin{array}{ll} I(P(a,a)) = \mathbf{t}, & I(P(a,b)) = \mathbf{t} \\ I(P(b,b)) = \mathbf{t}, & I(P(b,a)) = \mathbf{f} \\ I(Q(a)) = \mathbf{f}, & I(Q(b)) = \mathbf{t} \end{array}$$

study and motivate if I *is a model of* $\neg P(X,b) \vee Q(a)$ *and of* $P(Y,b) \vee Q(Y)$*.*

Exercise 50 (06/1997) *1. Study, from the semantic tree, if the following clause set is satisfiable:*

$$\{\neg Q(x_1), \neg P(x_2,b) \vee Q(a), P(x_3,a)\}$$

2. *write down the formulæ of a deductive structure whose clause set is the same as above. According to the above result, would the deduction be correct?*

Exercise 51 (02/1998) *Consider*

$$\begin{array}{l} C_1 : P(a,x,a,a) \\ C_2 : \neg N(x,y) \vee \neg P(r,y,s,z) \vee P(f(x,r),y,f(x,s),z) \\ C_3 : \neg M(x,y) \vee \neg P(r,y,z,s) \vee P(f(x,r),y,z,f(x,s)) \end{array}$$

1. *compute the Herbrand universe corresponding to* $\{C_1, C_2, C_3\}$*;*
2. *specify six element of the Herbrand base of* $\{C_1, C_2, C_3\}$*.*

Exercise 52 (09/1998) *Let* $\{A_1, A_2, A_3, A_4, A_5\}$ *the following clause set:*

$$\begin{array}{l} A_1 : \forall x \exists y (A(x,y) \to B(x) \wedge C(y)) \\ A_2 : \neg \forall y C(y) \\ A_3 : \forall x (B(x) \to \exists x \exists y \neg D(x,y)) \\ A_4 : \forall x \neg E(x) \\ A_5 : \forall x \forall y D(x,y) \vee \exists x E(x) \end{array}$$

Starting from the clause set, verify (and motivate) with a countermodel, obtained with a semantic tree, that it is not possible to deduce $\forall x \neg D(x,x)$ *from* $\{A_4, A_5\}$*.*

Exercise 53 (09/1998) *Let $\{A_1, A_2, A_3, A_4\}$ be the following set of formulæ:*

$$A_1 : \exists x B(x)$$
$$A_2 : \forall x \forall y (C(x,y) \rightarrow D(x) \wedge B(x))$$
$$A_3 : \exists x \forall y C(x,y)$$
$$A_4 : \neg \exists x \exists y (D(x) \wedge \neg A(y))$$

1. *find, by means of a semantic tree, a finite set of ground instances of clauses which allow to state that from $\{A_1, A_2, A_3, A_4\}$ we can deduce $\exists x B(x) \wedge \forall x A(x)$.*
2. *what can we say about the satisfiability of the above instance set?*
3. *verify by means of a semantic tree, and motivate with a countermodel, that it is not possible to deduce $\exists x \neg B(x)$ from $\{A_1, A_3\}$.*

Exercise 54 (06/1998) *Let $\{A_1, A_2, A_3, A_4\}$ be the following set:*

$$A_1 : R(b) \wedge G(b) \wedge R(a)$$
$$A_2 : \forall x (M(x) \rightarrow (G(x) \rightarrow P(x)) \wedge (\neg G(x) \rightarrow \neg P(x)))$$
$$A_3 : \forall x (R(x) \rightarrow M(x))$$
$$A_4 : \neg \exists x (R(x) \wedge \neg D(x))$$

Prove and motivate by means of a semantic tree that $G(a)$ cannot be deduced from the statement which is obtained by removing all reference to the constant symbol b (or clauses containing b in the above set).

Exercise 55 (02/1999) *Let $\{A_1, A_2, A_3\}$ be a set of formulæ:*

$$A_1 : \exists y \forall x (A(x,y) \rightarrow B(x) \wedge D(x,y))$$
$$A_2 : \forall x (B(x) \rightarrow \exists y \neg D(y,x) \vee E(x))$$
$$A_3 : \forall x \neg E(x)$$

Verify by means of a semantic tree, and motivate with a model of the corresponding set of clauses, that it is not possible to deduce $\forall x \exists y (\neg D(x,y) \vee B(y) \vee A(x,y))$ from $\{A_1\}$.

Exercise 56 (09/1999) *Given the clauses:*

$$C_1 : r(x) \vee p(x) \vee \neg q(h(x))$$
$$C_2 : \neg r(x)$$
$$C_3 : p(y) \vee \neg s(y, h(y)) \vee r(y)$$
$$C_4 : \neg s(z,x)$$
$$C_5 : q(y) \vee r(y)$$
$$C_6 : s(f(x), x) \vee \neg p(f(x))$$

prove that they are unsatisfiable, by using a semantic tree (and motivate why they are, according to the Herbrand theorem).

Exercise 57 (09/2000) *Consider a theory whose non-logical axioms are as follows:*

$$F_1 : \forall x (A(x) \rightarrow D(x) \wedge E(x))$$
$$F_2 : \neg \forall x \exists y (A(x) \wedge B(x) \rightarrow \neg A(y))$$
$$F_3 : \forall x (D(x) \rightarrow (B(x) \leftrightarrow C(x)))$$

Verify by means of a semantic tree that the formula $\exists x \neg (C(x) \wedge E(x))$ is not a theorem of the above theory, and motivate the result.

Exercise 58 (09/2001) *Given the following set of clauses:*

$$\{p(f(a)) \vee q(x), p(x) \vee q(g(x)), \neg q(a)\}$$

identify:

1. *the Herbrand universe and the Herbrand base;*
2. *one Herbrand interpretation;*
3. *one Herbrand interpretation which assign to the clauses the same truth value as the following interpretation I on natural numbers:*
   - $I(a) = 0$

- $f_I(x) = 2x + 1, \qquad g_I(x) = 2x$
- $p_I(x) \Leftrightarrow x$ *is odd*
- $q_I(x) \Leftrightarrow x$ *is even (0 is not even)*

4. *all the ground instances of clauses belonging to the original set which can be obtained by combining literals from the following set:*

$$\{p(f(a)), q(a), q(h(a))\}$$

Exercise 59 (09/2002) *Given the following set of formulæ:*

$$\begin{array}{l} F_1 : \forall x(q(x) \rightarrow \exists y \neg r(x,y) \vee s(x)) \\ F_2 : \exists y \forall x(p(x,y) \rightarrow q(x) \wedge r(y,x)) \\ F_3 : \forall x \forall y p(x,y) \\ F_4 : \forall x \neg s(x) \end{array}$$

prove, by means of a semantic tree and motivating with the Herbrand theorem, that the set is unsatisfiable.

Solution 14 *See Luís Iraola's resolved exercises.*

Exercise 60 (09/2003) *Given the following set of clauses:*

$$\{M(x) \vee \neg G(x), \neg M(x) \vee R(x), \neg R(x) \vee \neg M(x)\}$$

1. *build a semantic tree and specify which of the corresponding interpretations are models, if any;*
2. *what can be said about the satisfiability of the above set?;*
3. *build a semantic tree for the above clauses, plos $G(b)$; study if the new set is satisfiable.*

Exercise 61 (02/2003) *Given the following set of clauses:*

$$\begin{array}{l} F_1 : \forall x(A(x) \wedge \exists y \neg B(y) \rightarrow C(x,y)) \\ F_2 : \exists x A(x) \wedge \neg \forall y \exists z C(z,y) \\ F_3 : \forall y \exists x \forall z((B(x) \rightarrow A(z)) \wedge (\neg C(y,z) \rightarrow \neg B(y))) \end{array}$$

study if it is satisfiable; motivate your answer.

Exercise 62 (06/2003) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : P(f(x)) \vee \neg R(a,y) \vee S(x) \\ C_2 : Q(x,b) \vee \neg P(x) \\ C_3 : \neg Q(x,y) \vee S(y) \\ C_4 : R(x,y) \end{array}$$

1. *find, by means of a semantic tree, a finite set of ground instances which makes it possible to prove that $\exists x S(x)$ can be deduced from $\{C_1, C_2, C_3, C_4\}$.*
2. *verify, and motivate by showing a suitable interpretation, that $S(a)$ cannot be deduced from the same set of clauses.*

Solution 15 *See Luís Iraola's resolved exercises.*

Exercise 63 (09/2004) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : P(x) \vee P(y) \\ C_2 : \neg Q(a,y) \\ C_3 : \neg R(x) \vee \neg P(y) \\ C_4 : R(y) \vee Q(b,y) \end{array}$$

1. *compute the Herbrand universe and the Herbrand base;*
2. *find, by means of a semantic tree, a model of the clause set; what can be said about its satisfiability?*
3. *prove, by a semantic tree, that $\exists x \forall y \neg Q(x,y)$ can be deduced from the above clauses.*

Exercise 64 (06/2004) *Given the following set of clauses:*

$$\begin{array}{l} A_1 : \neg P(x) \vee T(x) \\ A_2 : P(m) \\ A_3 : Q(a,b) \vee \neg P(a) \\ A_4 : \neg Q(x,b) \\ A_5 : \neg T(c) \end{array}$$

1. *compute the Herbrand base and two Herbrand interpretations;*
2. *for each of the such interpretations, specify two instances of the above clauses which are satisfied by it;*
3. *is* $\{A_2, A_3, A_4\}$ *satisfiable? Motivate.*

Exercise 65 (06/2005) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : \neg C(x) \vee \neg B(x,y) \vee D(x) \\ C_2 : \neg A(x) \vee \neg B(x,y) \vee \neg D(y) \\ C_3 : \neg A(x) \vee C(x) \\ C_4 : A(f(x)) \\ C_5 : B(x, f(x)) \end{array}$$

1. *prove by means of a semantic tree that, from the clause set* $\{C_1, C_2, C_3, C_4, C_5\}$*, it is possible to deduce* $\forall x \neg A(x)$.
2. *is there a set of ground instances which guarantees the same result? which one?.*
3. *explain the result of the previous step by giving the corresponding theorem.*
4. *is the result obtained in the previous steps actually meaningful? (that is, are we really proving a non-trivial theorem?) Give a motivated answer starting from an analysis of the premises.*

Exercise 66 (06/2006) *Given the following set of clauses:*

$$\{p(f(a)) \vee \neg q(x), \neg p(a)\}$$

1. *define an Herbrand interpretation which assign to the clauses the same truth value as the following interpretation* I *on natural numbers (without 0):*
   - $I(a) = 1$
   - $f_I(x) = x + 1$
   - $p_I(x) \Leftrightarrow x$ *is smaller than 3*
   - $q_I(x) \Leftrightarrow x$ *is greater than or equal to 5*
2. *is the above interpretation a model or a countermodel of the clause set? motivate.*

Exercise 67 (09/2006) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : A(y,x) \vee \neg B(x,y) \\ C_2 : B(a,x) \\ C_3 : \neg C(y) \vee \neg A(a,y) \end{array}$$

1. *prove by means of a semantic tree that the deduction of* $\exists x \forall y (C(y) \rightarrow \neg B(y,x))$ *from* $\{C_1, C_2, C_3\}$ *is correct;*
2. *prove with the semantic tree that the set* $\{C_1, C_2, C_3\}$ *is satisfiable.*

Exercise 68 (06/2006) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : R(a,x) \\ C_2 : Q(g(y),y) \vee \neg R(y, f(y)) \\ C_3 : \neg Q(g(z),u) \vee P(f(u)) \end{array}$$

1. *prove by means of a semantic tree that the formula* $\exists x P(x)$ *can be deduced from* $\{C_1, C_2, C_3\}$.
2. *give an Herbrand interpretation which shows that* $P(a)$ *cannot be deduced from* $\{C_1, C_2, C_3\}$.

Exercise 69 (06/2007) 1. *Give a semantic tree which proves that the following set of proposition is satisfiable; motivate your answer;*

$$\{\neg p \vee \neg q, q \vee r, \neg r \vee \neg q, p \vee r\}$$

2. *give an interpretation which is a model of the above set; explain how such interpretation can be obtained from the above semantic tree;*
3. *define a relation between the above set and the following set of ground instances, such that it will be possible to apply to the latter the same semantic tree; what can be deduced about the set of ground instances?*

$$\begin{array}{c} \{\neg A(f(a),b) \vee \neg C(a, f(b)), \neg A(f(a),b) \vee \neg B(g(f(a)), g(a)), \\ A(f(a),b) \vee C(a, f(b)), C(a, f(b)) \vee B(g(f(a)), g(a))\} \end{array}$$

Solution 16 *See Luís Iraola's resolved exercises.*

Exercise 70 (02/2008) *Given the following set of clauses C:*

$$C = \{Q(x,y), \neg P(x,y), R(a,y) \lor P(x,y), \neg R(f(x),x) \lor \neg Q(f(x),x)\}$$

1. *define an Herbrand interpretation which is a model of C;*
2. *define an Herbrand interpretation which is a countermodel of C.*

Exercise 71 *Given the clause set S:*

$$\{\neg P(x) \lor Q(x,y), \neg Q(a,f(z)) \lor P(f(z)), P(a), \neg P(f(u))\}$$

1. *write down the associated semantic tree;*
2. *is it closed? why?*
3. *give a finite and unsatisfiable set of clauses, ground instances of clauses belonging to S.*

Solution 17 *See Luís Iraola's resolved exercises.*

Exercise 72 *Is the Herbrand universe of a formula finite? countable? uncountable? prove your guess.*

Exercise 73 *What is the necessary and sufficient condition for $H(F)$ to be finite?*

Exercise 74 *Is the Herbrand base of a formula finite? Countable? Uncountable? Prove your guess.*
What is the cardinality (i.e., the number of elements) of the Herbrand base w.r.t. the Herbrand universe?

Exercise 75 *Think about the definition of Herbrand Interpretation, where we say that I maps a constant to itself. What does it mean?*

Exercise 76 *Take the following formula F and put it in clause form.*

$$\forall x(r(x) \rightarrow (\exists y \exists z(p(y) \land p(z) \land q(y,z,x))))$$

Let f and g the names of the newly introduced Skolem functions. Then, for each of the following interpretations of F, find the corresponding Herbrand interpretations.

$I_1 : D_1 = \mathbf{N}$
- $f(x) =$ *the predecessor of x*
- $g(x) =$ *the integer division by 2 of x*
- $p(x)$ *means that x is prime*
- $q(x,y,z)$ *means that z is the sum of x and y*
- $r(x)$ *means that x is even and non-zero*

$I_2 : D_2 = \{0,1,2,3,4,5\}$
- $f(x) =$ *the successor of x*
- $g(0) = 1$
- $g(x) = x$*multiplied by 5, modulo 6 (if $x \neq 0$)*
- $p(x)$ *means that $x = 0$*
- $q(x,y,z)$ *means that z is x multiplied by y, modulo 6*
- $r(x)$ *means that $x \neq 0$*

Consider the domain and the interpretation of predicates in I_1. Do you recognize the meaning of F under I_1 (regardless of how f and g are interpreted)? Does I_1 satisfy F? What is the meaning of f and g in the clause form, considering that they were not in F?

Exercise 77 *Prove the following lemma: if an interpretation $\mathcal{I} = (D,I)$ satisfies F , then all Herbrand interpretations of F which correspond to $\mathcal{I}$ also satisfy F.*
Hint: first try the case where F has constants, then when F does not.

Exercise 78 *Find an example of a formula F and an interpretation I where I does not satisfy F but some corresponding I_H does.*
Note: it is possible to find one even when F has constants.

Exercise 79 *Find a rule for computing how many Herbrand interpretations correspond to a given (D,I), in the case where no constants appear in the formulæ.*

Exercise 80 *Consider the clause $C = p(x) \vee q(x, f(x))$ and the interpretation*

$$\begin{array}{rl} I_H = \{ & \neg p(a), \neg p(f(a)), \neg p(f(f(a))), \ldots \\ & \neg q(a,a), q(a, f(a)), \neg q(a, f(f(a))), \ldots \\ & \neg q(f(a), a), q(f(a), f(a)), \neg q(f(a), f(f(a))), \ldots \\ & \ldots \} \end{array}$$

1. *does I_H satisfy C?*
2. *try to find an interpretation I on* **N** *(natural numbers) such that I_H possibly corresponds to I and I satisfies C iff I_H does.*

Note: since the description of I_H is not complete, there could be many Is; just find a reasonable one.
Note: in order to do this exercise you somehow have to interpret the dots; just do it freely but consistently!

Exercise 81 *Consider the clauses $\mathcal{C} = \{p(x),\ q(f(f(y)))\}$ and the interpretation*

$$\begin{array}{rl} I_H = \{ & p(a), p(f(a)), p(f(f(a))), \ldots \\ & q(a), \neg q(f(a)), q(f(f(a))), q(f(f(f(a)))) \ldots \} \end{array}$$

(1) Does I_H satisfy C? (2) Try to find an interpretation I on **N** *(natural numbers) such that I_H possibly corresponds to I and I satisfies C iff I_H does.*
Note: since the enumeration of I_H is not complete, there could be many I. Just find a reasonable one.
Note: in order to do this exercise you somehow have to interpret the dots. Just do it freely but consistently!

Exercise 82 *Find an unsatisfiable set S of ground instances of clauses in $\mathcal{C}_i$, for*

1. $\mathcal{C}_1 = \{p(x, a, g(x, b)),\ \neg p(f(y), z, g(f(a), b))\}$
2. $\mathcal{C}_2 = \{p(x),\ q(x, f(x)) \vee \neg p(x),\ \neg q(g(y), z)\}$

Exercise 83 (02/2009) *Consider the following set of clauses:*

C_1: $\neg q(x, y) \vee r(x, b)$
C_2: $\neg r(x, z) \vee p(z)$
C_3: $q(g(x), a) \vee s(a, y) \vee p(x)$
C_4: $\neg s(z, y)$

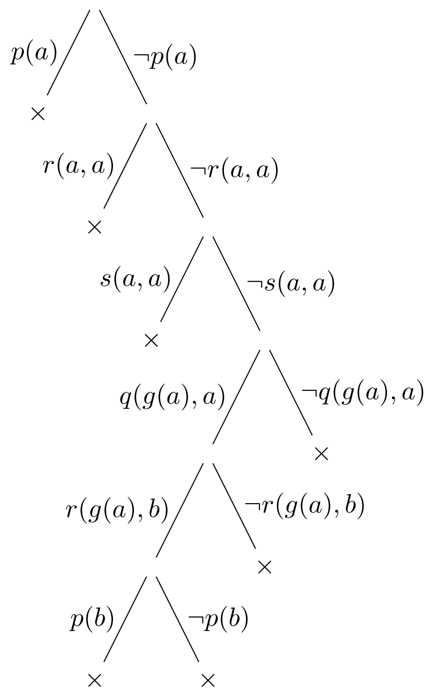
1. *prove, by using a semantic tree (after computing its Herbrand universe and its Herbrand base), that $\exists z p(z)$ can be deduced from $\{C_1, C_2, C_3, C_4\}$;*
2. *computing a closed complete semantic tree t implies identifying an unsatisfiable set C_t of ground clauses:*
   - *which proof uses this result? Write down the content of the theorem;*
   - *identify a possible C_t starting from the semantic tree you've just computed;*
   - *apply, if possible, the first three rules (in order) of the method of Davis-Putnam, and show the result on C_t (note: (1) apply rule 1 while possible; (2) then apply rule 2 while possible; (3) then apply rule 3 while possible).*

Solution 18

1. *Herbrand universe: $U = \{g^n(a) | n \geq 0\} \cup \{g^n(b) | n \geq 0\}$, where $g^0(x) \equiv x$ and $g^{n+1}(x) \equiv g(g^n(x))$. Herbrand base:*

$$B = \{p(t) | t \in U\} \cup \{q(t, t') | t, t' \in U\} \cup \{r(t, t') | t, t' \in U\} \cup \{s(t, t') | t, t' \in U\}$$

   A possible semantic tree (certainly, not the only one!):



2. *We are clearly talking about the Herbrand Theorem, which says that A set of clauses $\mathcal{C}$ is unsatisfiable iff there exists a finite set of ground instances of $\mathcal{C}$ clauses which is unsatisfiable. The proof of one direction of the theorem makes use the semantic tree, in the sense that we can pick a ground instance of some clause in $\mathcal{C}$ for every failure node of the tree, thus obtaining a set C_t. Since the tree is finite, all Herbrand interpretations falsify at least some instance in C_t, so that C_t is unsatisfiable.*
Therefore, the set C_t we are looking for consists in one ground instance for every failure node of the tree. A choice for C_t in our case is (ordered from the highest to the lowest failure node in the tree):

$$C_t = \{ \neg p(a),\ \neg r(a,a) \vee p(a),\ \neg s(a,a),\ q(g(a),a) \vee s(a,a) \vee p(a), \\ \neg q(g(a),a) \vee r(g(a),b),\ \neg p(b),\ \neg r(g(a),b) \vee p(b) \}$$

Now we want to apply the method of Davis-Putnam:
- *the tautology rule cannot be applied*
- *the one-literal can be applied until the empty clause is obtained:*

$$\{\neg p(a), \neg r(a,a) \vee p(a), \neg s(a,a), q(g(a),a) \vee s(a,a) \vee p(a), \\ \neg q(g(a),a) \vee r(g(a),b), \neg p(b), \neg r(g(a),b) \vee p(b)\}$$

$$[\neg p(a)]$$

$$\{\neg r(a,a), \neg s(a,a), q(g(a),a) \vee s(a,a), \\ \neg q(g(a),a) \vee r(g(a),b), \neg p(b), \neg r(g(a),b) \vee p(b)\}$$

$$[\neg r(a,a)]$$

$$\{\neg s(a,a), q(g(a),a) \vee s(a,a), \\ \neg q(g(a),a) \vee r(g(a),b), \neg p(b), \neg r(g(a),b) \vee p(b)\}$$

$$[\neg s(a,a)]$$

$$\{q(g(a),a), \neg q(g(a),a) \vee r(g(a),b), \neg p(b), \neg r(g(a),b) \vee p(b)\}$$

$$[q(g(a),a)]$$

$$\{r(g(a),b), \neg p(b), \neg r(g(a),b) \vee p(b)\}$$

$$(r(g(a),b))$$

$$\{\neg p(b), p(b)\}$$

$$[\neg p(b)]$$

$$\square$$

Note that the empty clause comes out because we are removing $\neg p(b)$ as a whole clause, but $p(b)$ as the last literal of a clause which becomes empty (and an empty disjunction is false)
- *therefore, no need to apply the pure-literal rule*

Exercise 84 (04/2009) *Consider the following clause set:*

$$\{p(a) \vee q(b), \neg p(a) \vee \neg q(b) \vee r(c), \neg r(c)\}$$

By using ground resolution, prove its unsatisfiability or find a Herbrand model.

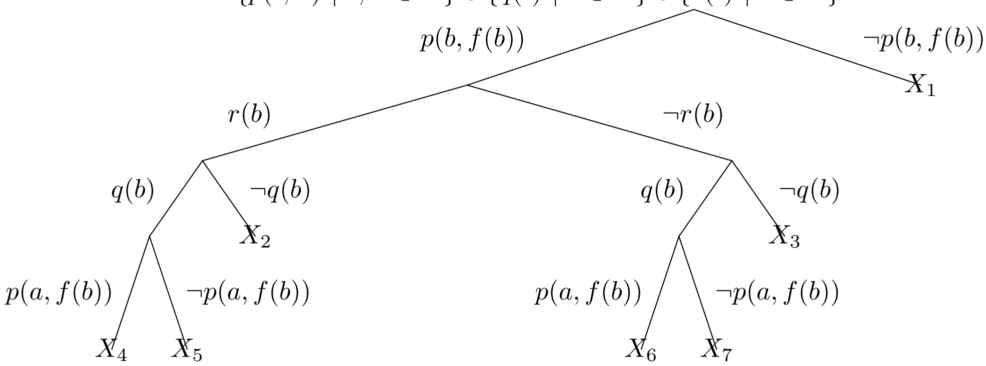
Exercise 85 (04/2009) *Given the following set $\mathcal{C}$ of clauses:*

C_1 $\neg p(x, f(x)) \vee q(x) \vee r(b)$
C_2 $\neg p(a, f(y)) \vee \neg q(y)$
C_3 $p(x, f(b))$
C_4 $\neg r(b) \vee q(b)$

1. *prove, by means of a semantic tree, that $\mathcal{C}$ is not satisfiable;*
2. *find a finite and unsatisfiable set of ground clauses of $\mathcal{C}$.*

Solution 19

Herbrand universe: $H = \{f^n(a) \mid n \geq 0\} \cup \{f^n(b) \mid n \geq 0\}$
Herbrand base: $B = \{p(t, t') \mid t, t' \in H\} \cup \{q(t) \mid t \in H\} \cup \{r(t) \mid t \in H\}$



- X_1 *falsifies the instance* $p(b, f(b))$ *of* C_3
- X_2 *falsifies* C_4
- X_3 *falsifies the instance* $\neg p(b, f(b)) \vee q(b) \vee r(b)$ *of* C_1
- X_4 *y* X_6 *falsify the instance* $\neg p(a, f(b)) \vee \neg q(b)$ *of* C_2
- X_5 *y* X_7 *falsify the instance* $p(a, f(b))$ *of* C_3

An unsatisfiable set of clauses is $\{p(b, f(b)), \neg r(b) \vee q(b), \neg p(b, f(b)) \vee q(b) \vee r(b), \neg p(a, f(b)) \vee \neg q(b), p(a, f(b))\}$

Exercise 86 (02/2008) *Given the following set C of clauses:*

$$C = \{Q(x, y), \neg P(x, y), R(a, y) \vee P(x, y), \neg R(f(x), x) \vee \neg Q(f(x), x)\}$$

1. *define a Herbrand interpretation which is a model for C;*
2. *define a Herbrand interpretation which is a countermodel for C.*

Exercise 87 (02/2008) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : \neg A(x, y) \vee \neg P(x) \vee Q(x) \\ C_2 : \neg A(x, y) \vee \neg R(f(x)) \vee \neg Q(y) \\ C_3 : P(x) \vee \neg R(y) \\ C_4 : R(f(x)) \vee R(f(y)) \\ C_5 : A(x, f(x)) \end{array}$$

1. *demonstrate using a semantic tree that this set of clauses is unsatisfiable;*
2. *define a set of basic instances of the clauses that justify the previous conclusion according to the Herbrand theorem.*

Exercise 88 (02/2006) *Given the set of formulae $\{F_1, F_2, F_3\}$ such that*

$$\begin{array}{l} F_1 : \exists y \forall x (A(x, y) \rightarrow B(x) \wedge D(x, y)) \\ F_2 : \forall x (B(x) \rightarrow \exists y \neg D(x, y) \vee E(x)) \\ F_3 : \forall x \neg E(x) \end{array}$$

1. *define the clause form for the given formulæ;*
2. *prove by means of a semantic tree that $\forall x \exists y (\neg D(x, y) \vee B(y) \vee A(x, y)))$ can not be deduced from $\{F_1\}$. Justify this conclusion with a model (an interpretation that satisfies the set of clauses) for the corresponding clause form.*

Exercise 89 (06/2006) *Given the following set of clauses:*

$$S = \{\neg p(b) \vee q(y), \neg q(x) \vee p(y), \neg r(y) \vee q(a)\}$$

1. *define an Herbrand interpretation which is a model of S;*
2. *define an Herbrand interpretation which is not a model of S.*

Exercise 90 (02/2007) *Let $[P_1, ..., P_n] \vdash Q$ be a deductive structure, C be its clause form, and I be an interpretation for C in any domain D.*

1. *if $[P_1, ..., P_n] \vdash Q$ is correct, what can be said about the satisfiability of C?*
2. *if a set of basic instances of clauses in C is satisfiable, what can be stated about $[P_1, ..., P_n] \vdash Q$?*
3. *in order to state that $[P_1, ..., P_n] \vdash Q$ is not correct, what condition must the corresponding semantic tree show?*
4. *what is an Herbrand interpretation for C and what is its relationship with the semantic tree for C?*

Exercise 91 (02/2007) *Given the following set of clauses:*

$$\begin{aligned}
&C_1 : \neg B(x) \vee \neg C(x, f(x)) \vee D(x)\\
&C_2 : \neg A(x, a) \vee B(x)\\
&C_3 : \neg A(x, a) \vee C(a, x)\\
&C_4 : A(x, y)\\
&C_5 : \neg D(x)
\end{aligned}$$

1. *demonstrate, using a semantic tree, that this set of clauses is unsatisfiable;*
2. *justify your answer using the Herbrand Theorem.*

Exercise 92 (09/2008) *Define an unsatisfiable set of basic instances for the next set of clauses and justify the unsatisfiability.*

$$\{A(x, y) \vee A(y, x), \neg A(y, f(x)) \vee \neg A(z, z) \vee B(x), \neg B(z)\}$$

Exercise 93 (09/2008) *Given the following set of clauses:*

$$\begin{aligned}
&C_1 : \neg P(x, a) \vee Q(x)\\
&C_2 : \neg Q(y) \vee P(b, y)\\
&C_3 : \neg P(a, x) \vee \neg Q(y)\\
&C_4 : P(x, b)
\end{aligned}$$

1. *define the Herbrand universe and the Herbrand base;*
2. *analyse, using a semantic tree, if the set is satisfiable or unsatisfiable;*
3. *define two Herbrand interpretations, a model and a countermodel, for the set of clauses.*

Exercise 94 (04/2009) *Given the following clauses:*

$$\begin{aligned}
&C_1 : Q(g(y), y) \vee \neg R(y, f(y))\\
&C_2 : R(a, x)\\
&C_3 : \neg Q(g(z), u) \vee P(f(u))
\end{aligned}$$

1. *define an Herbrand interpretation which proves that $P(a)$ cannot be deduced from $\{C_1, C_2, C_3\}$. To this end, define the Herbrand universe ans the Herbrand base;*
2. *prove, by means of a semantic tree, that $\exists x P(x)$ can be deduced from $\{C_1, C_2, C_3\}$;*
3. *considering the semantic tree above, apply Herbrand theorem to prove that $\{C_1, C_2, C_3\} \cup \{\neg P(x)\}$ is unsatisfiable.*

Solution 20

1.

$$H = \{a, f^n(a), g^n(a), f^n(g^m(a)), g^n(f^m(a))\}$$

$$BH = \{P(t_1), Q(t_1, t_2), R(t_1, t_2)\}_{t_1, t_2 \in H}$$

$$IH(P(a)) = \mathbf{f}, \quad IH(P(f(t))) = \mathbf{t}, \quad IH(R(a, t)) = \mathbf{t}, \quad IH(Q(g(t), t)) = \mathbf{t}$$

The rest of the atoms can either be interpreted as $\mathbf{t}$ or $\mathbf{f}$. This interpretation verifies $\{C_1, C_2, C_3\}$ and falsifies $P(a)$ at the same time.

2. *Let $C_4 : \neg P(x)$ the clause obtained by negating $\exists x P(x)$. A closed semantic tree is the following:*
   - *Level 1: $P(f(a)) = \mathbf{t}$; failure in C_4 - $\{x/a\} \rightarrow \neg P(f(a))$*
   - *Level 1: $P(f(a)) = \mathbf{f}$*
   - *Level 2: $R(a, f(a)) = \mathbf{t}$*
   - *Level 2: $R(a, f(a)) = \mathbf{f}$; failure in C_2 - $\{x/a\} \rightarrow R(a, f(a))$*
   - *Level 3: $Q(g(a), a)) = \mathbf{t}$; failure in C_3 - $\{z/a, u/a\} \rightarrow \neg Q(g(a), a) \vee P(f(a))$*
   - *Level 3: $Q(g(a), a)) = \mathbf{f}$; failure in C_1 - $\{y/a\} \rightarrow Q(g(a), a) \vee \neg R(a, f(a))$*

   Since this tree is closed, there does not exist an Herbrand interpretation which satisfies $\{C_1, C_2, C_3, C_4\}$; therefore, this set is unsatisfiable.
3. *The set of the clauses which label failure nodes, namely,*

$$\{\neg P(f(a)),\ R(a, f(a)),\ \neg Q(g(a), a) \vee P(f(a)),\ Q(g(a), a) \vee \neg R(a, f(a))\}$$

   (a) is finite; (b) is unsatisfiable; (c) contains only clauses which are ground instances of the set $\{C_1, C_2, C_3, C_4\}$. Therefore, by the Herbrand Theorem, $\{C_1, C_2, C_3, C_4\}$ is unsatisfiable.

Exercise 95 (07/2009) *Given the following set C of clauses:*

$$\begin{array}{ll} C_1 : \neg p(x, f(b)) \vee q(x) & C_5 : \neg t(z) \\ C_2 : s(b, b) & C_6 : r(f(v)) \vee \neg p(y, f(y)) \\ C_3 : \neg r(a) & C_7 : s(v, w) \vee t(b) \\ C_4 : \neg q(x) \vee \neg s(g(x), h(x)) & \end{array}$$

1. *prove that such set is unsatisfiable by means of an Herbrand interpretation which is a model of C;*
2. *find an Herbrand interpretation which is a countermodel of C;*
3. *prove, by means of a semantic tree, that $\exists z(s(b, z) \wedge \neg p(b, f(z)))$ can be deduced from C;*
4. *find a substitution which, if applied to C_5, makes the previous deduction (step (3)) impossible;*
5. *consider the set of ground clauses extracted from the semantic tree at (3): is it satisfiable? because of which result? write the result and justify its application to the present case.*

Solution 21 1. *in order to prove satisfiablity, it is enough to give an Herbrand model; for example:*

$$\forall x \forall y(s(x, y) = \mathbf{t}), \forall x(r(x) = \mathbf{f}), \forall x(t(x) = \mathbf{f}), \forall x \forall y(p(x, y) = \mathbf{f}), \forall x(q(x) = \mathbf{f})$$

   which means that, for example, s is true for every x and y, r is false for every x, etc. Such interpretation can be defined from the Herbrand universe H and the Herbrand base B:

$$H = \{a, b, f(a), f(b), g(a), g(b), h(a), h(b),$$
$$f(f(a)), f(f(b)), f(g(a)), f(g(b)), f(h(a)), f(h(b)), g(f(a)), g(f(b)), g(g(a)), g(g(b)), \ldots\}$$

$$B = \{p(d', d''), q(d), r(d), s(d', d''), t(d) | \forall d, d', d'' \in H\}$$

   The Herbrand interpretation I, which is a model, can be written as:

$$I = \{s(d', d''), \neg r(d), \neg t(d), \neg p(d', d''), \neg t(d) | \forall d, d', d'' \in H\}$$

2. *any interpretation which makes true, for example, $r(a)$ is a countermodel: $I = \{\ldots, r(a), \ldots\}$. Therefore, $\{A | A \in B\}$ (i.e., no negated atoms) is a countermodel. Clearly, there are much more;*
3. *as usual, we have to negate the conclusion and compute the clause form of the negation (in this order). The cause form of $\neg\exists z(s(b, z) \wedge \neg p(b, f(z)))$ is*

$$C_0 = \neg s(b, z) \vee p(b, f(z))$$

   the following step is to compute the semantic tree of $\{C_0, .., C_7\}$ and see that it is closed...
4. *in order for the set $\{C_0, ..., C_7\}$ to be unsatisfiable (i.e., to have a correct deduction), we need the "fact" $\neg t(b)$, which is an instance of (i.e., is implied by) C_5; if we replace C_5 by an instance of its which does NOT imply $\neg t(b)$, then the clause set becomes satisfiable. For example, $\{z/a\}$ is a substitution which generates $C'_5 = \neg t(a)$, and makes $\{C_0, C_1, C_2, C_3, C_4, C'_5, C_6, C_7\}$ satisfiable (because $\neg t(a)$ does NOT imply $\neg t(b)$). Therefore, C_0 cannot be deduced from $\{C_1, C_2, C_3, C_4, C'_5, C_6, C_7\}$.*

5. *the closed and complete semantic tree of step (3) identifies an unsatisfiable set of ground instances: those which become false at failure nodes.*

$$\{\neg t(b), s(g(b), h(b)) \vee t(b), \neg q(b) \vee \neg s(g(b), h(b)), \neg p(b, f(b)) \vee q(b), \neg s(b, b) \vee p(b, f(b)), s(b, b)\}$$

   For every $\delta \in D$, let $n(\delta)$ be the corresponding failure node (where it becomes false). We know that such set is unsatisfiable since, by definition of semantic tree and of failure node, every instance $\delta \in D$ becomes false in EVERY interpretation whose representation is a path in the tree which goes through $n(\delta)$. Moreover, being the tree closed, there are no interpretations which does not go through any failure node, i.e., which does not falsify any $\delta \in D$. Therefore, no interpretation is a model, so that S is unsatisfiable.

Exercise 96 (05/2010) *Let A, B, C, D, E, F, G formulædefined in a first-order language. The following information is available about them:*

- *A is a tautology and B is unsatisfiable*
- *C is satisfiable and D is the negation of C*
- *nothing is known about E*
- *the specific interpretation I is a countermodel of F and G*

For every sentence, say YES, NO, or UNK (if nothing can be said):

1. *$F \rightarrow G$ is satisfiable*
2. *$A \wedge C$ is satisfiable*
3. *$A \wedge C$ is true for the specific interpretation I*
4. *$D \rightarrow C$ is true for the specific interpretation I*
5. *$D \vee C$ is true for the specific interpretation I*
6. *$A \wedge B$ is satisfiable*
7. *$A \wedge E$ is satisfiable*
8. *$D \wedge C$ is true for the specific interpretation I*
9. *$A \wedge C \rightarrow B$ is unsatisfiable*

4 "Ground" methods for automatic proofs

Exercise 97 (06/2000) *Given the following clause set*

$$C = \{\neg q(a, x) \vee r(y), \quad p(y), \quad \neg r(b), \quad \neg p(f(x)) \vee q(z, f(x))\}$$

and the set of ground instances

$$I = \{\neg q(a, f(a)) \vee r(b), \quad p(f(a)), \quad \neg r(b), \quad \neg p(f(a)) \vee q(b, f(a))\}$$

study the satisfiability of I by using the method of Gilmore. What can be said about the satisfiability of C?

Exercise 98 (06/2001) *Given the following set of ground clauses:*

$$\left\{ \begin{array}{c} \neg q(a) \vee r(c),\ p(b) \vee r(c) \vee \neg p(b),\ p(b) \vee \neg r(a), \\ \neg r(b) \vee p(a),\ p(a) \vee q(a) \vee r(c), r(a) \vee \neg p(b),\ p(b) \vee \neg r(c) \vee p(a), \\ p(c) \vee \neg r(b),\ r(a) \vee p(b) \vee \neg p(a),\ \neg q(a), \neg p(b) \vee \neg p(a) \end{array} \right\}$$

A *identify tautologies, one-literals and pure literals;*
B *give the result of applying the rules of the method of Davis-Putnam which refer to what identified in (A);*
1. *apply the splitting rule on $r(a)$ to the set produced in (B);*
2. *go on applying the method of Davis-Putnam to the result of step (C);*
3. *relying on the result of step (D), tell if the original set is satisfiable, and motivate.*

Exercise 99 (02/2003) *Given the following clauses:*

$$\begin{array}{l} C_1 : \neg c \vee q \\ C_2 : \neg b \vee c \vee \neg d \\ C_3 : q \vee b \vee c \\ C_4 : d \vee q \end{array}$$

1. *write the clauses in implicative form (i.e., with at least one implication connective);*
2. *write the following resolvents:*
   R_1 : *resolvent between* C_2, C_3
   R_2 : *resolvent between* R_1, C_1
   R_3 : *resolvent between* C_2, C_4
   R_4 : *resolvent between* R_2, C_4
   R_5 : *resolvent between* C_1, C_3
   R_6 : *resolvent between* R_5, R_3
   R_7 : *resolvent between* R_6, C_1
3. *with the initial set of clauses and the resolvent obtained in the previous step, write a linear resolution where only such clauses are used, and which allows to prove that q can be deduced;*
4. *the same as the previous step, but with a non-linear resolution.*

Exercise 100 1. *Apply Gilmore's method to the following formulæ, and try to decide their satisfiability:*
   (a) $(\neg p \vee q) \wedge \neg q \wedge p$
   (b) $(p \vee q) \wedge (r \vee q) \wedge \neg r \wedge \neg q$
   (c) $p \wedge q \wedge r$
   (d) $(p \vee q) \wedge (\neg p \vee q) \wedge r$
   (e) $(p \vee q) \wedge \neg q$
   (f) $(p \vee q) \wedge (\neg p \vee q) \wedge (\neg r \vee \neg q) \wedge (r \vee \neg q)$
2. *Apply the method of Davis-Putnam to the six formulæ above.*

Exercise 101 *Consider the following chemical reactions:*

$$\begin{array}{c} MgO + H_2 \rightarrow Mg + H_2O \\ C + O_2 \rightarrow CO_2 \\ CO_2 + H_2O \rightarrow H_2CO_3 \end{array}$$

and suppose there is a sufficient amount of MgO, H_2, O_2, and C. Show, by means of the method of Davis-Putnam, that we can make H_2CO_3.

Exercise 102 *Prove, by Davis-Putnam, that the following formula is valid:*

$$(((q \to p) \land (p \to q)) \to (\neg q \land \neg r)) \lor (((r \to p) \land (q \to s)) \to ((p \to r) \to (r \land s)))$$

Exercise 103 *Prove the following set of clauses is unsatisfiable by ground resolution:*

$$\{p \lor q \lor r,\ \neg p \lor r,\ \neg q,\ \neg r\}$$

Exercise 104 *For the set* $S = \{p \lor q, \neg q \lor r, \neg p \lor q, \neg r\}$ *derive an empty clause from* S *by ground resolution.*

Exercise 105 (04/2009) *Given the following set of clauses:*

$$\begin{gathered}\{p(a,c) \lor \neg q(b) \lor p(a,a), \neg q(b) \lor r(a) \lor \neg p(a,c), \neg p(c,a) \lor q(c) \lor r(b),\\ \neg p(a,b) \lor q(a), p(a,c), q(b) \lor p(a,a), \neg p(a,a) \lor \neg r(a) \lor p(a,b),\\ p(c,a) \lor \neg r(b) \lor q(c), \neg r(a) \lor \neg q(a), q(b) \lor \neg p(a,a), \neg q(b) \lor p(a,a) \lor \neg r(a)\}\end{gathered}$$

1. *apply the following steps of the method of Davis-Putnam: one-literal rule, pure-literal rule, and splitting rule on* $q(b)$.
2. *Determine the satisfaibility of the two sets obtained by the splitting rule, and the satisfiability of the initial set.*

Exercise 106 (05/2009) *Given the following set of clauses:*

$$\begin{gathered}\{ \quad t(c) \lor s(c,c), \quad \neg s(c,c) \lor \neg t(c), \quad \neg s(a,c), \quad t(a) \lor \neg s(a,a) \lor r(f(c)),\\ s(c,c) \lor \neg t(e), \quad r(a) \lor \neg s(a,c), \quad \neg s(e,e) \lor t(c), \quad \neg r(f(c)) \lor t(e),\\ r(a) \lor s(e,e) \lor s(a,c) \lor \neg s(c,c) \lor \neg r(c), \quad r(c) \lor \neg r(a) \lor t(e), \quad t(e) \lor \neg t(c) \lor s(a,c), \}\end{gathered}$$

1. *apply the one-literal rule, deleting the affected literals;*
2. *if necessary, apply the pure-literal rule, deleting the affected literals;*
3. *if necessary, apply the splitting rule on the remaining literals, taking* $t(c)$ *as the splitting atom; let* S' *and* S'' *the resulting sets;*
4. *study the satisfiability of* S' *and* S'' *(by using Davis-Putnam or any other suitable method) and, consequently, of the initial set.*

Solution 22 *After applying one-literal:*

$$\begin{gathered}\{ \quad t(c) \lor s(c,c), \quad \neg s(c,c) \lor \neg t(c), \quad t(a) \lor \neg s(a,a) \lor r(f(c)),\\ s(c,c) \lor \neg t(e), \quad \neg s(e,e) \lor t(c), \quad \neg r(f(c)) \lor t(e),\\ r(a) \lor s(e,e) \lor \neg s(c,c) \lor \neg r(c), \quad r(c) \lor \neg r(a) \lor t(e), \quad t(e) \lor \neg t(c) \ \}\end{gathered}$$

After applying pure-literal:

$$\begin{gathered}\{ \ t(c) \lor s(c,c), \quad \neg s(c,c) \lor \neg t(c), \quad s(c,c) \lor \neg t(e), \quad \neg s(e,e) \lor t(c),\\ r(a) \lor s(e,e) \lor \neg s(c,c) \lor \neg r(c), \quad r(c) \lor \neg r(a) \lor t(e), \quad t(e) \lor \neg t(c) \ \}\end{gathered}$$

(note that $\neg r(f(c))$ *becomes pure after the first application of the rule)*

Splitting on $t(c)$*, we obtain*

$$S' = \{ \ s(c,c), \quad s(c,c) \lor \neg t(e), \quad \neg s(e,e), \quad r(a) \lor s(e,e) \lor \neg s(c,c) \lor \neg r(c), \quad r(c) \lor \neg r(a) \lor t(e), \quad \}$$

$$S'' = \{ \ \neg s(c,c), \quad s(c,c) \lor \neg t(e), \quad r(a) \lor s(e,e) \lor \neg s(c,c) \lor \neg r(c), \quad r(c) \lor \neg r(a) \lor t(e), \quad t(e) \ \}$$

Clearly, S' *is satisfiable, but* S'' *is not. Therefore, the initial set is satisfiable.*

Exercise 107 (05/2009) *Consider the following clause set:*

$$\begin{aligned}&C_1 : p(a) \lor q(f(a))\\ &C_2 : \neg r(x) \lor \neg q(x) \lor r(f(a))\\ &C_3 : \neg r(a) \lor \neg q(y)\\ &C_4 : p(a) \lor r(a)\\ &C_5 : \neg p(z)\end{aligned}$$

Study if it is satisfiable by using ground resolution *with* level saturation.

Solution 23 *With the first level $H_0 = \{a\}$ of the Herbrand universe, we have the following ground clauses:*

$$\begin{array}{l} I_1 : p(a) \vee q(f(a)) \\ I_2 : \neg r(a) \vee \neg q(a) \vee r(f(a)) \\ I_3 : \neg r(a) \vee \neg q(a) \\ I_4 : p(a) \vee r(a) \\ I_5 : \neg p(a) \end{array}$$

No contradiction can be found by resolution. In fact, there is a model: the one where $p(a) = \mathbf{f}$, $q(f(a)) = \mathbf{t}$, $r(a) = \mathbf{t}$, $q(a) = \mathbf{f}$ and $r(f(a))$ is free.
Then, we have to go to $H_1 = \{a, f(a)\}$. In this case, the instance set is:

$$\begin{array}{l} I_1 : p(a) \vee q(f(a)) \\ I_2^1 : \neg r(a) \vee \neg q(a) \vee r(f(a)) \\ I_2^2 : \neg r(f(a)) \vee \neg q(f(a)) \vee r(f(a)) \\ I_3^1 : \neg r(a) \vee \neg q(a) \\ I_3^2 : \neg r(a) \vee \neg q(f(a)) \\ I_4 : p(a) \vee r(a) \\ I_5^1 : \neg p(a) \\ I_5^2 : \neg p(f(a)) \end{array}$$

We can obtain by resolution:

$$\begin{array}{ll} R_1 : p(a) \vee \neg q(f(a)) & [I_3^2, I_4] \\ R_2 : \neg q(f(a)) & [R_1, I_5^1] \\ R_3 : p(a) & [I_1, R_2] \\ R_4 : \Box & [R_3, I_5^1] \end{array}$$

Therefore, the set is unsatisfiable, and the initial set also is.

Exercise 108 (05/2009) *Given the following clause set:*

$$\begin{array}{c} \{ \quad \neg q(b) \vee r(a) \vee \neg p(a,c), \quad p(a,c) \vee \neg q(b) \vee p(a,a), \quad q(c) \vee r(b) \vee \neg p(c,a), \\ p(a,a) \vee q(b), \quad \neg p(a,b) \vee q(a), \quad p(c,a) \vee q(c) \vee \neg r(b), \quad \neg p(a,a) \vee \neg r(a) \vee p(a,b), \quad p(a,c), \\ \neg r(a) \vee \neg q(a), \quad q(b) \vee \neg p(a,a), \quad \neg q(b) \vee \neg r(a) \vee p(a,a) \quad \} \end{array}$$

1. *apply the one-literal rule, and delete the affected literals;*
2. *if necessary, apply the pure-literal rule, and delete the affected literals;*
3. *if necessary, apply the splitting rule on $p(a,a)$; let S' and S'' be the resulting sets;*
4. *Study the satisfiability of S' and S'' (using any method you know) and, therefore, the satisfiability of the initial set*

Exercise 109 (05/2009) *Consider the following set of ground clauses:*

$$\{\neg r(e), \quad p(a) \vee q(c), \quad \neg p(a) \vee q(c) \vee r(e)\}$$

Using resolution, prove the unsatisfiability or, if not possible, find a Herbrand interpretation which is a model.

Exercise 110 (05/2009) *Consider the following set of ground clauses:*

$$\begin{array}{l} C_1 : \neg s(z) \\ C_2 : s(a) \vee t(f(a)) \\ C_3 : \neg r(x) \vee \neg t(x) \vee r(f(a)) \\ C_4 : r(a) \vee s(a) \\ C_5 : \neg r(a) \vee \neg t(y) \end{array}$$

Study if it is satisfiable by using ground resolution *with* level saturation.

Exercise 111 (05/2009) *Given the following clause set:*

$$\begin{array}{c} \{ \, p(a,c) \vee \neg q(b) \vee p(a,a), \quad \neg q(b) \vee r(a) \vee \neg p(a,c), \quad \neg p(c,a) \vee q(c) \vee r(b), \quad \neg p(a,b) \vee q(a), \quad p(a,c), \\ q(b) \vee p(a,a), \quad \neg p(a,a) \vee \neg r(a) \vee p(a,b), \quad p(c,a) \vee \neg r(b) \vee q(c), \\ \neg r(a) \vee \neg q(a), \quad q(b) \vee \neg p(a,a), \quad \neg q(b) \vee p(a,a) \vee \neg r(a) \quad \} \end{array}$$

1. *apply the one-literal rule, and delete the affected literals;*

2. *if necessary, apply the pure-literal rule, and delete the affected literals;*
3. *if necessary, apply the splitting rule on* $p(a, a)$*; let* S' *and* S'' *be the resulting sets;*
4. *Study the satisfiability of* S' *and* S'' *(using any method you know) and, therefore, the satisfiability of the initial set*

Exercise 112 (05/2009) *Given the following clause set:*

$$\{p(a) \vee q(b), \quad \neg p(a) \vee \neg q(b) \vee r(c), \quad \neg r(c)\}$$

Using resolution, prove the unsatisfiability or, if not possible, find a Herbrand interpretation which is a model.

Exercise 113 (05/2009) *Given the following clause set:*

$$\{p(a) \vee q(b), \quad \neg r(b) \vee \neg q(b) \vee r(c), \quad \neg r(c) \vee \neg q(b), \quad p(a) \vee r(b), \quad \neg p(a)\}$$

Compute the resolvent which results from applying resolution to the following pairs of clauses:

- $p(a) \vee q(b), \qquad \neg p(a)$
- $\neg r(b) \vee \neg q(b) \vee r(c), \qquad \neg r(c) \vee \neg q(b)$

Using resolution, prove the unsatisfiability or, if not possible, find a Herbrand interpretation which is a model.

5 Unification

Exercise 114 (09/1994) *Does the UMG of the following pairs of formulæ exist? Motivate the answer, and write both the UMG and the atomic formula obtained by applying it.*

- $C(x, a, g(x)), C(b, y, z)$
- $B(y, f(x)), B(g(z), z)$

Exercise 115 (06/1999) *Compute if the following pair of atomic formulæ are unifiable. If they are, specify the MGU and the formula resulting from applying it:*

- $R(f(x), f(x)), R(y, f(y))$
- $T(u, f(x), x), T(g(z), z, a)$
- $R(a, x), R(b, y)$

Exercise 116 (09/1999) *Given the following formulæ:*

$$F_1 : \forall x(\forall y(A(x, y) \vee B(y)) \rightarrow C(x) \vee D(x))$$
$$F_2 : \forall x \exists y(C(y) \rightarrow A(y, x))$$

Resolve every clause obtained from the clause form of F_1 with every clause from the clause form of F_2, in every possible way. Specify which literals it is possible to resolve on, and obtain the resulting resolvents and the MGU in every step.

Exercise 117 (09/2001) *Find the MGU of the following pairs, if it exists, or show why it does not exist:*

- $r(g(x), h(g(a))), \qquad r(y, h(y))$
- $r(f(a), h(b)), \qquad r(f(z), h(z))$
- $s(x, f(g(x), y), g(y), a), \qquad s(b, f(z, w), z, w)$
- $t(x, f(y, x), g(y)), \qquad t(v, f(w, g(w)), g(v))$

Exercise 118 (06/2001) *Find, if it exists, the MGU of the following pairs of literals.*

- $q(a, y, g(a, y)), \qquad q(z, x, g(z, f(x)))$
- $p(g(a), y, f(y), u), \qquad p(x, f(x), z, g(z))$

Exercise 119 (09/2002) *If possible, find the MGU of the following pairs of atomic formulæ; if not possible, explain why.*

- $P(f(x, y), g(y), a), \qquad P(f(t, z), t, z)$
- $Q(f(x), a, x), \qquad Q(f(g(y)), y, z)$
- $R(x, a, f(x, y)), \qquad R(g(z), t, f(z, b))$

Exercise 120 (02/2003) *Find, if it exists, the MGU of the following pairs. Show how to compute the MGU, and the result of applying it to both formulæ, or motivate why it cannot be found.*

- $Q(a, g(y, a), y), \qquad Q(z, g(f(x), x), x)$
- $R(h(f(y)), v, b), \qquad R(h(w), g(x, w), x)$

Exercise 121 (06/2004) *Study if the following pairs are unifiable.*

- $P(h(x), g(x, z), z), \qquad P(h(t), g(s, h(s)), t)$
- $Q(f(g(e)), g(z)), \qquad Q(f(x), x)$

Exercise 122 *For each of the following pairs of atomic formulæ, determine if they are unifiable; if they are, then provide tha MGU; if they are not, then say why:*

- $F(a, f(x)), \qquad F(y, b)$
- $H(f(a), g(a, y)), \qquad H(f(x), g(x, b))$
- $P(x, x, y), \qquad P(z, f(a), f(b))$
- $R(f(x), x), \qquad R(y, f(y))$

Exercise 123 (02/2006) *For each of the following pairs of atomic formulæ, determine if they are unifiable; if they are, then provide tha MGU; if they are not, then say why:*

– $P(z, x, f(h(x, z)))$, $\quad P(a, y, f(y))$
– $Q(x, f(a), g(g(y)))$, $\quad Q(z, f(z), g(y))$
– $R(z, f(y), g(y))$, $\quad R(x, x, z)$

Exercise 124 *Let*

$$\alpha = \{\ x/a,\ y/f(b),\ z/c\ \} \qquad \beta = \{\ v/f(f(a)),\ z/x,\ x/g(y)\ \}$$

– *compute* $\alpha\beta$ *and* $\beta\alpha$
– *for every of the following formulæ, compute (i)* $F\alpha$*; (ii)* $F\beta$*; (iii)* $F\alpha\beta$*; (iv)* $F\beta\alpha$
  1. $p(x, y, z)$;
  2. $p(h(v)) \vee \neg q(z, x)$
  3. $q(x, z, v) \vee \neg q(g(y), x, f(f(a)))$
– *are* α, β, $\alpha\beta$, $\beta\alpha$ *idempotent?*

Exercise 125 *For every* C_1, C_2, *and* α, *decide whether (i)* α *is a unifier of* C_1 *and* C_2*; and (ii)* α *is the MGU of* C_1 *and* C_2.

C_1	C_2	α
$p(a, f(y), z)$	$q(x, f(f(v)), b)$	$\{\ x/a,\ y/f(b),\ z/b\ \}$
$q(x, h(a, z), f(x))$	$q(g(g(v)), y, f(w))$	$\{\ x/g(g(v)),\ y/h(a, z),\ w/x\ \}$
$q(x, h(a, z), f(x))$	$q(g(g(v)), y, f(w))$	$\{\ x/g(g(v)),\ y/h(a, z),\ w/g(g(v))\ \}$
$r(f(x), g(y))$	$r(z, g(v))$	$\{\ x/a,\ z/f(a),\ y/v\ \}$

Exercise 126 *Find, when possible, the MGU of the following pairs of clauses.*

$\{q(a),\ q(b)\}$
$\{q(a, x),\ q(a, a)\}$
$\{q(a, x, f(x)),\ q(a, y, y)\}$
$\{q(x, y, z),\ q(u, h(v, v), u)\}$
$\{p(x_1, g(x_1), x_2, h(x_1, x_2), x_3, k(x_1, x_2, x_3)),\ p(y_1, y_2, e(y_2), y_3, f(y_2, y_3), y_4)\}$

Exercise 127 (02/2009)

1. *by following the unification algorithm, find, if possible, the MGU of these literals; show how the algorithm works;*

	L'	–	L''
A:	$p(g(x, f(z)), b, y)$	–	$p(g(a, x), y, f(f(x)))$
B:	$q(a, g(h(x)), x, f(f(h(c))))$	–	$q(z, g(y), b, f(f(y)))$

2. *apply the rule of factorization to the following clauses:*
   A: $p(x, a) \vee q(b, b, g(x)) \vee p(f(z), y)$
   B: $r(y, a) \vee r(f(z), z) \vee s(y)$

Solution 24

1. *MGUs (primed variables refer to* L'*, double-primed variables refer to* L''*):*

$L'\alpha$	$L''\alpha$	t'	t''
$p(g(x', f(z')), b, y')$	$p(g(a, x''), y'', f(f(x'')))$	x'	a
			$\alpha = \{x'/a\}$
$p(g(a, f(z')), b, y')$	$p(g(a, x''), y'', f(f(x'')))$	$f(z')$	x''
			$\alpha = \{x'/a,\ x''/f(z')\}$
$p(g(a, f(z')), b, y')$	$p(g(a, f(z')), y'', f(f(f(z'))))$	b	y''
			$\alpha = \{x'/a,\ x''/f(z'),\ y''/b\}$
$p(g(a, f(z')), b, y')$	$p(g(a, f(z')), b, f(f(f(z'))))$	y'	$f(f(f(z')))$
			$\alpha = \{x'/a,\ x''/f(z'),\ y''/b,\ y'/f(f(f(z')))\}$
$p(g(a, f(z')), b, f(f(f(z'))))$	$p(g(a, f(z')), b, f(f(f(z'))))$		

$L'\alpha$	$L''\alpha$	t'	t''
$q(a, g(h(x')), x', f(f(h(c))))$	$q(z'', g(y''), b, f(f(y'')))$	a	z''
			$\alpha = \{z''/a\}$
$q(a, g(h(x')), x', f(f(h(c))))$	$q(a, g(y''), b, f(f(y'')))$	$h(x')$	y''
			$\alpha = \{z''/a,\ y''/h(x')\}$
$q(a, g(h(x')), x', f(f(h(c))))$	$q(a, g(h(x')), b, f(f(h(x'))))$	x'	b
			$\alpha = \{z''/a,\ y''/h(b),\ x'/b\}$
$q(a, g(h(b)), b, f(f(h(c))))$	$q(a, g(h(b)), b, f(f(h(b))))$	c	b
			fail

2. *Factorized clauses:*

	C'	α
A	$p(f(z), a) \vee q(b, b, g(f(z)))$	$\{x/f(z),\ y/a\}$
B	$r(f(a), a) \vee s(f(a))$	$\{y/f(a),\ z/a\}$

Exercise 128 (05/2009) *For the following pairs of atomic formulæ, find, if it exists, the MGU. Explain the process.*

$$P(x, f(g(u), u), g(x)) \qquad P(h(y), f(z, h(a)), z)$$
$$Q(g(x, y), h(x, y)) \qquad Q(g(z, u), h(w, u)) \qquad Q(g(t, t), h(v, f(v)))$$

Exercise 129 (06/2009) *For the following pairs of atoms, find the MGU, if it exists. Apply the well-known algorithm, showing each step in the table below.*

$(t_a,\ t_b)$	*New binding*	*Unifier*	*Formulæ obtained by applying the unifier*
/////	/////	λ	$p(a, w, g(v, v), h(w, w)), \quad p(z, h(b, f(y)), g(y, x), y)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$(t_a,\ t_b)$	*New binding*	*Unifier*	*Formulæ obtained by applying the unifier*
/////	/////	λ	$q(h(y), f(z, h(a)), z), \quad q(x, f(g(u), u), g(x))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Solution 25

		λ	$p(a,\ w,\ g(v, v),\ h(w, w)),$ $p(z,\ h(b, f(y)),\ g(y, x),\ y)$
a, z	z/a	$\{z/a\}$	$p(a,\ w,\ g(v, v),\ h(w, w)),$ $p(a,\ h(b, f(y)),\ g(y, x),\ y)$
$w, h(b, f(y))$	$w/h(b, f(y))$	$\{z/a, w/h(b, f(y))\}$	$p(a,\ h(b, f(y)),\ g(v, v),\ h(h(b, f(y)), h(b, f(y)))),$ $p(a,\ h(b, f(y)),\ g(y, x),\ y)$
v, y	v/y	$\{z/a, w/h(b, f(y)),$ $v/y\}$	$p(a,\ h(b, f(y)),\ g(y, y),\ h(h(b, f(y)), h(b, f(y)))),$ $p(a,\ h(b, f(y)),\ g(y, x),\ y)$
y, x	y/x	$\{z/a, w/h(b, f(x)),$ $v/x, y/x\}$	$p(a,\ h(b, f(x)),\ g(x, x),\ h(h(b, f(x)), h(b, f(x)))),$ $p(a,\ h(b, f(x)),\ g(x, x),\ x)$
$h(h(b, f(x)), h(b, f(x))), x$	*failure:*	*t_b appears in t_a*	

		λ	$q(h(y), f(z, h(a)), z)$, $q(x, f(g(u), u), g(x))$
$h(y), x$	$x/h(y)$	$\{x/h(y)\}$	$q(h(y), f(z, h(a)), z)$, $q(h(y), f(g(u), u), g(h(y)))$
$z, g(u)$	$z/g(u)$	$\{x/h(y), z/g(u)\}$	$q(h(y), f(g(u), h(a)), g(u))$, $q(h(y), f(g(u), u), g(h(y)))$
$h(a), u$	$u/h(a)$	$\{x/h(y), z/g(h(a)), u/h(a)\}$	$q(h(y), f(g(h(a)), h(a)), g(h(a)))$, $q(h(y), f(g(h(a)), h(a)), g(h(y)))$
a, y	y/a	$\{x/h(a), z/g(h(a)), u/h(a), y/a\}$	$q(h(a), f(g(h(a)), h(a)), g(h(a)))$, $q(h(a), f(g(h(a)), h(a)), g(h(a)))$

Exercise 130 (02/2006) *(a) Formally define Herbrand Theorem, and explain its role in the use of methods like Gilmore and Davis-Putnam. (b) Is there a most general unifier (MGU) for the following pairs of atomic formulæ? Motivate the answer, and define the MGU whenever possible.*

1. $P(z, x, f(h(x, z))), \quad P(a, y, f(y))$
2. $Q(x, f(a), g(g(y))), \quad Q(z, f(z), g(y))$
3. $R(z, f(y), g(y)), \quad R(x, x, z)$

Exercise 131 (02/2007) *Define, when possible, the most general unifier (MGU) for the following pairs of literals. If not possible, then justify your answer.*

(a) $R(h(a, x), a, x), \quad R(h(y, f(z)), y, z)$
(b) $P(u, g(y), y, f(a)), \quad P(f(z), z, g(x), x)$

Exercise 132 (09/2008) *Is there a most general unifier (MGU) for the following pairs of atomic formulæ? Justify your answer, and define the MGU whenever possible.*

$P(h(x), g(x, z), z), \quad P(h(t), g(s, h(s)), t)$
$R(f(x), a, x), \quad R(f(g(y)), y, z)$

Exercise 133 (07/2009) *For the following pairs of atoms, find the MGU if it exists; show how the algorithm work by using the table below (D is the set of discordances; α is the current substitution; β is the substitution containing the binding which solves the discordance).*

(a) $p(f(x, g(x)), h(y), t) \quad p(y, h(t), f(g(z), w))$
(b) $p(x, a, g(x), b) \quad p(f(z, y), z, g(f(t, w)), t)$

Solution 26

(a)

β	α	D
$-$	λ	$\{(f(x, g(x)), y), (y, t), (t, f(g(z), w))\}$
$\{y/f(x, g(x))\}$	$\{y/f(x, g(x))\}$	$\{(f(x, g(x)), t), (t, f(g(z), w))\}$
$\{t/f(x, g(x))\}$	$\{y/f(x, g(x)), t/f(x, g(x))\}$	$\{(f(x, g(x)), f(g(z), w))\}$
$\{x/g(z)\}$ $\{w/g(g(z))\}$	$\{y/f(g(z)), g(g(z))), t/f(g(z), g(g(z))), x/g(z)\}$ $\{y/f(g(z)), g(g(z))), t/f(g(z), g(g(z))), x/g(z), w/g(g(z))\}$	$\{(f(g(z), g(g(z))), f(g(z), w))\}$

Formula obtained by applying the MGU: $p(f(g(z), g(g(z))), h(f(g(z))), f(g(z), g(g(z))))$ *(b)*

β	α	D
$-$	λ	$\{(x, f(z, y)), (a, z), (x, f(t, w)), (b, t)\}$
$\{x/f(z, y)\}$	$\{x/f(z, y)\}$	$\{(a, z), (f(z, y), f(t, w)), (b, t)\}$
$\{z/a\}$	$\{x/f(a, y), z/a\}$	$\{(f(a, y), f(t, w)), (b, t)\}$
$\{t/a\}$	$\{x/f(a, y), z/a, t/a\}$	$\{(f(a, y), f(a, w)), (b, a)\}$

The discordance (b, a) cannot be solved (both are different constants): there is no MGU.

6 MGU resolution

Exercise 134 (06/1988) *Study, with the method of resolution, if* $T[F_1, F_2, F_3, F_4] \vdash B$*, being:*

$$\begin{aligned} F_1 &: \exists x P(x) \\ F_2 &: \forall x R(x) \\ F_3 &: \forall x(\forall y(R(y) \wedge P(x)) \rightarrow Q(s(x))) \\ F_4 &: \forall x(Q(x) \rightarrow P(s(x))) \\ B &: \exists x P(s(s(s(s(x))))) \end{aligned}$$

with x*,* y *variable symbols,* s *unary function symbol,* P*,* Q*,* R *unary predicate symbols. Repeat the analysis with* $B = \forall x P(x)$*.*

Exercise 135 (09/1989) *Prove, by resolution, the unsatisfiability of the following set of clauses:*

$$\begin{aligned} C_1 &: M(a, f(c), f(b)) \\ C_2 &: M(x, x, f(x)) \\ C_3 &: \neg M(x, y, z) \vee M(y, x, z) \\ C_4 &: \neg M(x, y, z) \vee N(x, z) \\ C_5 &: P(a) \\ C_6 &: \neg N(a, b) \\ C_7 &: \neg M(y, z, u) \vee \neg P(x) \vee \neg N(x, u) \vee N(x, y) \vee N(x, z) \end{aligned}$$

Exercise 136 *Verify by resolution the validity of* $T[A_1, A_2, A_3] \vdash Q$*, being:*

$$\begin{aligned} A_1 &: \exists x P(x) \rightarrow \forall x Q(x) \\ A_2 &: \exists x \forall z(\neg \exists y R(x, y, z) \wedge P(z) \wedge L(x)) \\ A_3 &: \forall x \exists z((\neg T(x) \vee \exists y R(x, y, z)) \wedge (\neg L(z) \vee \neg T(x))) \\ Q &: \exists x \exists y \neg (Q(x) \rightarrow T(y)) \end{aligned}$$

Exercise 137 (02/1989) *Study, by using resolution, if* $T[F_1, F_2, F_3, F_4, F_5, F_6] \vdash G$*, being:*

$$\begin{aligned} F_1 &: B(c) \wedge P(a) \\ F_2 &: \forall x(R(x) \rightarrow O(x, c) \vee L(x, c)) \\ F_3 &: A(a, c) \\ F_4 &: \forall x(P(x) \rightarrow R(x)) \\ F_5 &: \forall x \exists y L(x, y) \\ F_6 &: \forall x \forall y(\neg A(x, y) \vee \neg B(y) \vee \neg L(x, y)) \\ G &: \exists x O(x, c) \end{aligned}$$

Exercise 138 (02/1990) *Verify by resolution that the following set of formulæis unsatisfiable:*

$$\begin{aligned} F_1 &: \forall x \exists y(E(x) \wedge \neg V(x) \rightarrow S(x, y) \wedge C(y)) \\ F_2 &: \exists x(P(x) \wedge E(x) \wedge \forall y(S(x, y) \rightarrow P(y))) \\ F_3 &: \forall x \forall y(P(x) \rightarrow \neg(C(x) \vee V(x))) \end{aligned}$$

Exercise 139 (09/1990) *Study, by using resolution, if* $T[F_1, F_2, F_3] \vdash G$*, being:*

$$\begin{aligned} F_1 &: \forall x \neg \exists y(\neg H(x, y) \wedge P(y) \wedge \neg T(y)) \\ F_2 &: \forall x \exists y \forall z(A(x, y) \wedge \neg T(y) \wedge (H(z, y) \vee T(x))) \\ F_3 &: \forall x \neg P(x) \\ G &: \exists x \exists y \neg (A(x, y) \wedge H(x, y) \rightarrow P(y)) \end{aligned}$$

Exercise 140 (06/1991) *Study, by using resolution, if* $T[A_1, A_2] \vdash B$*, being:*

$$\begin{aligned} A_1 &: \forall x(P(x) \rightarrow \exists y Q(x, y)) \\ A_2 &: \forall x \forall y(Q(x, y) \rightarrow P(f(y))) \\ B &: \forall x(P(x) \rightarrow \exists y P(f(y))) \end{aligned}$$

Exercise 141 (09/1991) *Study, by using resolution, if* $T[A_1, A_2] \vdash B$*, being:*

$$\begin{aligned} A_1 &: \forall x \exists y(A(x, y) \vee B(x, y)) \rightarrow \exists x \forall y C(x, y) \\ A_2 &: \forall x D(x) \rightarrow \forall x \forall y A(x, y) \\ B &: \exists x(\exists y(D(x) \wedge B(x, y)) \rightarrow \forall y C(x, y)) \end{aligned}$$

Exercise 142 (02/1992) *Study, by using resolution, if* $T[A_1, A_2, A_3] \vdash B$ *being:*

$$
\begin{aligned}
A_1 &: \forall x \forall y (R(y) \vee \neg P(x,y) \to \neg A(x)) \\
A_2 &: \exists x \forall y (A(x) \wedge (P(x,y) \to B(x))) \\
A_3 &: \forall x \forall y (\neg B(x) \vee (R(y) \vee C(y))) \\
B &: \exists x C(x)
\end{aligned}
$$

Exercise 143 (09/1992) *Given the following deductive strusture* $T[F_1, F_2, F_3, F_4] \vdash Q$*, with:*

$$
\begin{aligned}
F_1 &: \forall x \forall y (A(x,y) \to B(x,y) \wedge C(y)) \\
F_2 &: \exists x \forall y (D(x) \to \neg B(x,y)) \\
F_3 &: \exists x \forall y (C(x) \to D(y)) \\
F_4 &: \forall x \forall z (\forall y A(x,y) \to \exists y \neg E(z,y)) \\
Q &: \exists x \exists y (A(x,y) \to E(a,b))
\end{aligned}
$$

verify its validity by means of resolution, taking as the start clause some of the clauses obtained from the conclusion.

Exercise 144 (02/1992) *Study, by using resolution, if* $T[F_1, F_2, F_3] \vdash B$ *being:*

$$
\begin{aligned}
F_1 &: \forall x \exists y (A(x,y) \to \neg P(x)) \\
F_2 &: \exists x \forall y (\neg R(y,x) \vee P(x)) \\
F_3 &: \forall x \forall y (A(x,y) \wedge S(x)) \\
F_4 &: \exists x S(x) \\
B &: \forall x \exists y \neg R(x,y)
\end{aligned}
$$

Exercise 145 (09/1992) *Study, by using resolution, if* $T[F_1, F_2, F_3] \vdash G$ *being:*

$$
\begin{aligned}
F_1 &: \forall x (\neg B(x,x) \wedge D(x)) \\
F_2 &: \exists x (D(x) \to \forall y \neg B(x,y)) \\
F_3 &: \exists x (\neg A(x) \vee \neg E(x)) \\
G &: \forall x (A(x) \to E(x))
\end{aligned}
$$

Exercise 146 (09/1993) *Study, by using resolution, if* $T[] \vdash B$*, being* B*:*

$$\exists x \exists y (A(x) \wedge \neg B(x,y)) \vee \neg \forall x \exists y (B(x,y) \to C(y)) \vee \exists x C(x) \vee \exists x \forall y (A(x) \wedge B(x,y))$$

Exercise 147 (09/1993) *Given the set of fromulæ* $\{F_1, F_2, F_3, F_4\}$*:*

$$
\begin{aligned}
F_1 &: \forall z \exists x ((\neg P(z) \vee \exists y Q(x,y,z)) \wedge (\neg R(z) \vee \neg P(x))) \\
F_2 &: \exists z S(z) \to \forall z T(z) \\
F_3 &: \forall z \forall x (T(z) \to P(x)) \\
F_4 &: \exists z \forall y (\neg \exists x Q(z,x,y) \wedge S(y) \wedge R(z))
\end{aligned}
$$

Prove the unsatisfiability of this set by resolution.

Exercise 148 (02/1993) *Study, by using resolution, if* $T[F_1, F_2, F_3] \vdash G$ *being:*

$$
\begin{aligned}
F_1 &: \forall y (\neg \forall x (A(x) \wedge B(y)) \to C(a,y)) \\
F_2 &: \forall x \forall y (D(x,y) \to \neg C(x,y)) \\
F_3 &: A(a) \vee B(b) \to \forall x D(x,b) \\
G &: A(a) \leftrightarrow B(b)
\end{aligned}
$$

Exercise 149 (09/1994) *Is it possible to deduce the empty clause from the following clause sets?*

- $\{\neg B(x, f(y)), B(a,x) \vee C(x), B(b,x) \vee \neg C(x)\}$
- $\{A(x), \neg A(y) \vee B(a,y), \neg B(b,z)\}$

In case it is not, find an Herbrand interpretation which is a model of the set, by using a semantic tree.

Exercise 150 (02/1994) *Study, by using resolution, if* $T[A_1, A_2] \vdash B$*, being:*

$$
\begin{aligned}
A_1 &: \exists x \neg A(x) \\
A_2 &: \exists y \forall x (B(x,y) \to C(x)) \\
B &: \exists x C(x) \vee \forall x (B(x,x) \vee A(x))
\end{aligned}
$$

Exercise 151 (02/1994) *Apply resolution to deduce the empty clause from the following set:*

$$\begin{aligned}
&C_1 : P(x,a) \vee \neg R(b,x) \vee Q(x)\\
&C_2 : R(x,a) \vee \neg Q(x)\\
&C_3 : \neg P(f(x),x) \vee Q(x)\\
&C_4 : \neg Q(f(x))\\
&C_5 : R(x,y)\\
&C_6 : \neg Q(a)
\end{aligned}$$

Exercise 152 *Study, by resolution, if, from P_1, P_2, P_3, where*

$$\begin{aligned}
&P_1 : \forall x(A(x) \wedge B(x))\\
&P_2 : \forall x \neg A(x) \vee \exists y \forall x C(x,y)\\
&P_3 : \forall x \forall y \neg (C(x,y) \wedge D(x,y))
\end{aligned}$$

it is possible to obtain any of the following conclusions:

$$\begin{aligned}
&Q_1 : \forall x \exists y \neg D(x,y)\\
&Q_2 : \forall x \forall y (D(x,y) \rightarrow B(x))\\
&Q_3 : \exists x A(x) \rightarrow \neg \forall x \forall y D(x,y)
\end{aligned}$$

Exercise 153 (02/1995) *Considering the Axiom of Equality ($\forall x \forall y \forall z (y = z \rightarrow (S(x,y) \rightarrow S(x,z)))$), study if, from the formulæ $\forall x \exists y \exists z (F(x,y) \wedge S(x,z))$ and $\forall x \forall y (F(x,y) \rightarrow \neg S(x,y))$ it is possible to deduce $\exists x \exists y \neg (x = y)$, by using resolution.*

Exercise 154 (02/1995) *Study, by using resolution, if $T[A_1, A_2] \vdash B$, being:*

$$\begin{aligned}
&A_1 : \neg \exists x (A(a) \wedge C(x))\\
&A_2 : \exists x \forall y (\neg C(x) \rightarrow B(x,y))\\
&B : \forall x \exists y (\neg A(x) \vee B(x,y))
\end{aligned}$$

Exercise 155 (06/1996) *Study, by using resolution, if $T[A_1, A_2] \vdash B$, being:*

$$\begin{aligned}
&A_1 : \forall x \neg Q(x)\\
&A_2 : \forall x \exists y (P(x,y) \vee Q(y))\\
&B : \forall x \exists y P(x,y) \wedge \forall x Q(x)
\end{aligned}$$

Exercise 156 (09/1997) *Given the formulæ:*

$$\begin{aligned}
&P_1 : \forall x \forall y (M(x,y) \rightarrow Z(x,y))\\
&P_2 : \forall x \forall y (V(x) \rightarrow M(x,y))\\
&P_3 : \forall x \forall y \forall z (Z(x,y) \wedge Z(y,x) \rightarrow Z(x,z))\\
&P_4 : \forall x \forall y (L(x) \wedge B(y) \rightarrow Z(x,y))\\
&P_5 : \forall x \forall y (B(x) \wedge P(y) \rightarrow Z(x,y))\\
&P_6 : \forall x (P(x) \rightarrow V(x))\\
&Q : \forall x \forall y (\exists z B(z) \wedge L(x) \wedge P(y) \rightarrow Z(x,y))
\end{aligned}$$

1. *starting from the clause form of the deductive structure $T[P_1, P_2, P_3, P_4, P_5, P_6] \vdash Q$, apply resolution in order to determine if the empty clause can be obtained.*
2. *Is the above clause set unsatisfiable? Is $T[P_1, P_2, P_3, P_4, P_5, P_6] \vdash Q$ a correct deductive structure? Motivate the answers relying on the above result.*

Exercise 157 (09/1997) *Study, by using resolution, if $T[A_1, A_2] \vdash B$, being:*

$$\begin{aligned}
&A_1 : \neg \exists x \exists y (A(x,y) \wedge B(x) \wedge C(y))\\
&A_2 : \forall x (D(x) \rightarrow C(x))\\
&B : \forall x \forall y (A(x,y) \rightarrow \neg B(x) \vee \neg D(y))
\end{aligned}$$

Exercise 158 (02/1998) *Consider*

$$\begin{aligned}
&C_l : P(a,x,a,a)\\
&C_2 : P(f(x,r),y,f(x,s),z) \vee \neg N(x,y) \vee \neg P(r,y,s,z)\\
&C_3 : P(f(x,r),y,z,f(x,s)) \vee \neg M(x,y) \vee \neg P(r,y,z,s)
\end{aligned}$$

1. *deduce the empty clause by resolution, starting from:*

$$\{C_1, C_2, C_3\} \cup \{\neg P(f(1, f(3, a)), 2, x, y)\} \cup \{N(0,1), N(1,2), N(2,3)\} \\ \cup \{M(3,2), M(2,1), M(1,0)\}$$

2. *specify the values taken by x and y in the above derivation.*

Exercise 159 (09/1998) *Given the following clause set, where A, D, E are binary predictaes, B, C are unary predicates, f, g, h are unary functions, k is a binary function, and a is a constant symbol:*

$$\left\{ \begin{array}{c} \neg A(x, f(x)) \vee B(x), \neg A(x, f(x)) \vee C(f(x)), \neg B(g(x)) \vee \neg D(x, h(x)), \\ \neg E(x, y) \vee C(x), D(k(x, x), y) \vee E(x, y), \neg C(a) \end{array} \right\}$$

1. *prove by resolution that it is possible to deduce from these formulæ $\exists x \exists y \neg A(x, y)$;*
2. *which values of x and y allow to deduce $\neg A(x, y)$ in the previous step? Motivate.*

Exercise 160 (02/1998) *Apply, if possible, the factorization rule to:*

$$C_1 : \neg M(x, a) \vee \neg M(y, a) \vee P(f(x, y), a)$$
$$C_2 : \neg M(x, a) \vee M(a, y) \vee P(f(x, y), a)$$

Exercise 161 (09/1998) *Let $\{A_1, A_2, A_3, A_4, A_5\}$ the following set of formulæ:*

$$A_1 : \forall x \exists y (A(x, y) \rightarrow B(x) \wedge C(y))$$
$$A_2 : \neg \forall y C(y)$$
$$A_3 : \forall x (B(x) \rightarrow \exists x \exists y \neg D(x, y))$$
$$A_4 : \forall x \neg E(x)$$
$$A_5 : \forall x \forall y D(x, y) \vee \exists x E(x)$$

Study, by resolution, if, from $\{A_1, A_2, A_3, A_4, A_5\}$, it is possible to deduce $\exists x \exists y \neg A(x, y)$.

Exercise 162 (06/1998) *Let $\{A_1, A_2, A_3, A_4\}$ the following set of formulæ: siguiente:*

$$A_1 : R(b) \wedge G(b) \wedge R(a)$$
$$A_2 : \forall x (M(x) \rightarrow (G(x) \rightarrow P(x)) \wedge (\neg G(x) \rightarrow \neg P(x)))$$
$$A_3 : \forall x (R(x) \rightarrow M(x))$$
$$A_4 : \neg \exists x (R(x) \wedge \neg D(x))$$

Starting from the corresponding clause set, prove that $\exists x D(x) \wedge P(b)$ can be deduced from $\{A_1, A_2, A_3, A_4\}$.

Exercise 163 (02/1998) *Study, by using resolution, if $T[A_1, A_2, A_3] \vdash B$ being:*

$$A_1 : \exists x \forall y (P(x, y) \rightarrow \neg R(x) \wedge S(y))$$
$$A_2 : \forall x (S(x) \rightarrow R(x))$$
$$A_3 : \forall x \forall y (T(x, y) \rightarrow P(x, y))$$
$$B : \forall x \exists y \neg T(x, y)$$

Exercise 164 (02/1999) *Let $\{A_1, A_2, A_3\}$ the set of formulæ corresponding to:*

$$A_1 : \exists y \forall x (A(x, y) \rightarrow B(x) \wedge D(x, y))$$
$$A_2 : \forall x (B(x) \rightarrow \exists y \neg D(y, x) \vee E(x))$$
$$A_3 : \forall x \neg E(x)$$

Study, by resolution, that, from $\{A_1, A_2, A_3\}$, it is possible to deduce $\exists x \exists y \neg A(x, y)$. Specify the values of the variables in the formula which make the deduction possible.

Solution 27 *See Luís Iraola's resolved exercises.*

Exercise 165 (06/1999) 1. *Study, by resolution, if the following clause set C is unsatisfiable:*

$$C_0 : \neg p(x) \vee \neg r(x, y) \vee q(x)$$
$$C_1 : \neg d(x) \vee \neg r(x, y) \vee \neg q(y)$$
$$C_2 : \neg d(x) \vee p(x)$$
$$C_3 : d(f(x))$$
$$C_4 : d(a)$$
$$C_5 : r(x, f(x))$$

2. *If the predicate* $ans(x, y)$ *is added to* C_0*, which values do* x *and* y *take in the above resolution?*
3. *Find a finite and unsatisfiable set of ground instances of clauses from* C*.*

Exercise 166 (09/1999) *Given the following clause set:*

$$\begin{array}{l} C_1 : r(x) \lor p(x) \lor \neg q(h(x)) \\ C_2 : \neg r(x) \\ C_3 : p(y) \lor \neg s(y, h(y)) \lor r(y) \\ C_4 : \neg s(z, x) \\ C_5 : q(y) \lor r(y) \\ C_6 : s(f(x), x) \lor \neg p(f(x)) \end{array}$$

Prove that it is unsatisfiable, by using resolution (motivate).

Exercise 167 (06/2000) *Starting from the following set of clauses:*

$$\begin{array}{l} C_1 : \neg p(x, y, u) \lor \neg p(y, z, v) \lor \neg p(u, z, w) \lor p(x, v, w) \\ C_2 : \neg p(x, y, u) \lor \neg p(v, z, y) \lor \neg p(x, v, w) \lor p(u, z, w) \\ C_3 : p(x, e, x) \\ C_4 : p(x, i(x), e) \\ C_5 : p(i(x), x, e) \\ C_6 : \neg p(c, c, e) \end{array}$$

1. *give all resolvents of* $p(e, x, x)$ *with* C_1*, also considering clauses which are obtained from* C_1 *by factorization;*
2. *prove, by resolution, the unsatisfiability of the set;*
3. *specify a support set, and tell whether the previous refutation is directed according to such set.*

Exercise 168 (09/2000) 1. *Prove the unsatisfiability of the following clause set by means of linear resolution, starting from* C_0*:*

$$\begin{array}{l} C_0 : \neg P(f(y), a, x) \\ C_1 : P(x, y, z) \lor \neg Q(g(x), f(y), v) \\ C_2 : Q(x, x, y) \lor \neg M(z, v, v) \\ C_3 : Q(g(z), v, y) \lor \neg T(z, v, y) \\ C_4 : T(g(y), f(x), z) \lor \neg M(a, b, x) \\ C_5 : T(z, z, b) \lor \neg S(z, v, a) \\ C_6 : S(a, b, c) \\ C_7 : S(f(x), y, a) \lor \neg L(x, y, z) \\ C_8 : L(a, b, c) \\ C_9 : L(b, a, c) \end{array}$$

2. *reason about why the above derivation is linear;*
3. *is the above derivation input? Motivate.*

Exercise 169 (06/2001) *Write a non-linear refutation of the following clause set, motivate why it is not linear and specify when in the derivation the linearity property is broken.*

$$\begin{array}{l} C_1 : \neg A(x, y) \lor \neg P(x) \lor Q(x) \\ C_2 : \neg A(x, y) \lor \neg R(f(x)) \lor \neg Q(y) \\ C_3 : P(x) \lor \neg R(y) \\ C_4 : R(f(x)) \lor R(f(y)) \\ C_5 : A(x, f(x)) \end{array}$$

Exercise 170 (09/2001) *Study, by means of resolution, if the following set of clauses is unsatisfiable:*

$$\begin{array}{l} C_1 : \neg P(x, y) \lor \neg Q(f(y), x) \\ C_2 : P(x, g(x)) \\ C_3 : R(x, a) \lor \neg P(b, g(x)) \lor R(z, z) \\ C_4 : Q(f(g(x)), a) \lor \neg R(x, a) \lor \neg R(a, y) \\ C_5 : P(x, g(y)) \end{array}$$

Exercise 171 (06/2002) *Study, by directed resolution, if the following set of clauses is unsatisfiable, where the support set is* $\{C_2, C_3, C_4, C_5\}$:

$$\begin{array}{l} C_1 : \neg Q(g(y), z) \vee P(z) \\ C_2 : P(x) \vee Q(x, f(x)) \\ C_3 : \neg P(f(x)) \vee R(x) \\ C_4 : \neg R(x) \vee Q(x, z) \\ C_5 : \neg P(g(f(a))) \end{array}$$

Exercise 172 (09/2002) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : \neg A(x) \vee B(f(x), g(x), x) \\ C_2 : \neg D(y) \vee E(x) \\ C_3 : \neg C(z) \vee \neg A(f(z)) \\ C_4 : \neg E(x) \vee A(y) \\ C_5 : \neg B(a, x, y) \\ C_6 : C(a) \\ C_7 : D(y) \end{array}$$

1. *prove by resolution that the set is unsatisfiable;*
2. *is the above derivation linear? is it input? motivate your answer.*

Exercise 173 (06/2002) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : p(a, c) \vee p(b, c) \\ C_2 : p(a, d) \\ C_3 : \neg p(x, c) \vee p(e, x) \\ C_4 : p(x, y) \vee p(y, f(x)) \vee p(z, f(z)) \\ C_5 : \neg p(v, f(v)) \end{array}$$

1. *deduce* $\exists x p(e, x)$ *from the clauses* C_1, C_2, C_3; *obtain, by answer extraction on a linear resolution tree, the values of* x *for which it is possible to state* $p(e, x)$;
2. *obtain the MGU which allow to derive the clause* $p(f(x), f(x))$ *as the resolvent of* C_4 *and* C_5.

Exercise 174 (06/2003) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : R(y) \vee \neg Q(f(x)) \\ C_2 : \neg B(x) \vee R(y) \vee R(g(x, y)) \\ C_3 : B(x) \vee R(b) \vee Q(x) \vee H(b) \\ C_4 : R(y) \vee \neg H(b) \end{array}$$

write a resolution which allow to obtain the resolvent $R(b) \vee R(g(f(x), b))$.

Exercise 175 (06/2003) *Prove that the following set of clauses is unsatisfiable:*

$$\begin{array}{l} C_1 : A(x) \vee \neg B(g(x)) \vee C(x) \\ C_2 : \neg C(x) \\ C_3 : A(x) \vee C(x) \vee \neg D(x, g(x)) \\ C_4 : C(x) \vee \neg D(x, y) \\ C_5 : B(x) \vee C(x) \\ C_6 : \neg A(f(x)) \vee D(f(x), x) \end{array}$$

Exercise 176 (09/2003) *Find a non-linear refutation of the following set:*

$$\begin{array}{l} C_1 : A(x, f(x)) \\ C_2 : \neg B(x, y) \vee A(f(x), y) \\ C_3 : \neg C(f(x), y) \vee B(x, y) \\ C_4 : B(x, f(y)) \vee C(x, y) \\ C_5 : \neg A(x, x) \vee \neg B(y, x) \\ C_6 : \neg A(x, f(x)) \vee C(x, x) \vee D(x, x) \\ C_7 : \neg D(x, f(x)) \end{array}$$

Exercise 177 (09/2004) *Given the following set of clauses:*

$$
\begin{array}{l}
C_1 : \neg P(x) \vee Q(x) \vee R(f(x)) \\
C_2 : \neg P(x) \vee Q(x) \vee S(x, f(x)) \\
C_3 : \neg Q(x) \vee \neg T(x) \\
C_4 : \neg R(x) \vee \neg T(x) \\
C_5 : \neg S(a, x) \vee T(x) \\
C_6 : P(a) \\
C_7 : T(a)
\end{array}
$$

1. *build a resolution tree of the empty clause;*
2. *is such tree linear? Why?*

Exercise 178 (06/2004) *Verify by resolution that the following set of clauses is unsatisfiable. Motivate your answer.*

$$
\begin{array}{l}
C_1 : \neg P(x, y) \vee \neg A(x) \vee B(y) \\
C_2 : \neg P(x, y) \vee \neg D(x) \vee \neg B(f(y)) \\
C_3 : \neg D(x) \vee A(x) \\
C_4 : D(f(x)) \vee D(f(y)) \\
C_5 : P(x, f(x))
\end{array}
$$

Exercise 179 (09/2005) *Prove that the following set of clauses is unsatisfiable:*

$$
\begin{array}{l}
C_1 : A(x) \vee \neg B(h(x)) \vee C(x) \\
C_2 : \neg C(x) \\
C_3 : A(y) \vee C(y) \vee \neg D(y, h(y)) \\
C_4 : \neg D(x, y) \\
C_5 : B(y) \vee C(y) \\
C_6 : \neg A(f(x)) \vee D(f(x), x)
\end{array}
$$

Exercise 180 (02/2005) *Given the following set of clauses:*

$$
\begin{array}{l}
C_1 : \neg R(x) \vee S(x) \vee Q(x, g(x)) \\
C_2 : \neg R(x) \vee S(x) \vee T(g(x)) \\
C_3 : R(a) \\
C_4 : \neg Q(a, y) \vee P(y) \\
C_5 : \neg P(x) \vee \neg T(x) \\
C_6 : \neg P(f(y)) \vee \neg S(y)
\end{array}
$$

1. *is it correct to deduce $\exists x \neg P(f(x)) \wedge \exists y \neg P(y)$? motivate your answer by using resolution;*
2. *is the above resolution linear? is it input? motivate.*

Solution 28 *By negating $\exists x \neg P(f(x)) \wedge \exists y \neg P(y)$ and computing its clause form, we obtain $C_0 = P(f(x)) \vee P(y)$. In order to obtain the empty clause, we have to factorize C_0, which becomes $C_0' = P(f(x))$. Note that factorizing C_0 does not mean to replacing it by C_0'; rather, it means adding C_0' to the existing set, without dropping C_0.*

$$
\begin{array}{lcll}
R_1 & : & \neg S(y) & [C_0', C_6] \\
R_2 & : & \neg R(x) \vee T(g(x)) & [R_1, C_2] \\
R_3 & : & \neg R(x) \vee \neg P(g(x)) & [R_2, C_5] \\
R_4 & : & \neg P(g(a)) & [R_3, C_3] \\
R_5 & : & \neg Q(a, g(a)) & [R_4, C_4] \\
R_6 & : & \neg R(a) \vee S(a) & [R_5, C_1] \\
R_7 & : & S(a) & [R_6, C_3] \\
R_8 & : & \neg P(f(a)) & [R_7, C_6[\\
R_9 & : & \square & [R_8, C_0']
\end{array}
$$

This refutation is clearly linear and input.

Exercise 181 (06/2005) *Given the following set of clauses, obtain the empty clause by non-linear resolution.*

$$
\begin{array}{l}
C_1 : \neg P(x, y) \vee \neg Q(x, y) \vee \neg R(y) \\
C_2 : \neg P(x, y) \vee Q(y, x) \\
C_3 : \neg P(x, y) \vee S(y, x) \\
C_4 : R(x) \vee \neg S(x, x) \\
C_5 : P(x, f(x)) \\
C_6 : P(f(x), x) \\
C_7 : P(f(x), f(x))
\end{array}
$$

Exercise 182 (06/2007) *Prove by resolution that the following set of clauses is unsatisfiable:*

$$
\begin{array}{l}
C_1 : B(z, y) \vee \neg A(a, x) \vee C(f(y), g(x)) \\
C_2 : \neg C(y, g(z)) \vee B(y, z) \\
C_3 : \neg C(a, g(f(x))) \\
C_4 : \neg B(x, g(z)) \vee \neg B(f(y), y) \\
C_5 : A(x, y) \vee D(y, x) \\
C_6 : C(x, y) \vee \neg D(y, x)
\end{array}
$$

Exercise 183 *Determine whether the following clauses can be factorized, and give the factors if possible.*

1. $p(x) \vee q(y) \vee p(f(x))$
2. $p(x) \vee p(a) \vee q(f(x)) \vee q(f(a))$
3. $p(x, y) \vee p(a, f(a))$
4. $p(a) \vee p(b) \vee p(x)$
5. $p(x) \vee p(f(y)) \vee q(x, y)$

Exercise 184 *Find the possible resolvents of the following pairs of clauses.*

C	D
$\neg p(x) \vee q(x, b)$	$p(a) \vee q(a, b)$
$\neg p(x) \vee q(x, x)$	$\neg q(a, f(a))$
$\neg p(x, y, u) \vee \neg p(y, z, v) \vee \neg p(x, v, w) \vee p(u, z, w)$	$p(g(x, y), x, y)$
$\neg p(v, z, v) \vee p(w, z, w)$	$p(w, h(x, x), w)$

Exercise 185 *Apply resolution (with refutation) to prove that the following formula*

$$5 \quad m(5, f(7, f(5, f(1, 0))))$$

is a consequence of the set

$$
\begin{array}{ll}
1 & \neg m(x, 0) \\
2 & \neg i(x, y, z) \vee m(x, z) \\
3 & \neg m(x, z) \vee \neg i(v, z, y) \vee m(x, y) \\
4 & i(x, y, f(x, y))
\end{array}
$$

Note: $f(4, f(3, f(2, f(1, 0))))$ can be seen as the list $[4, 3, 2, 1, 0]$, so that m somehow represents list membership. Hint: remember to rename variables when necessary.

Exercise 186 *Consider the set of clauses*

$$\{ \neg r,\ r \vee \neg p \vee \neg q,\ p \vee \neg s \vee \neg t,\ p \vee \neg u,\ q,\ s,\ u \}$$

and give an SLD refutation in form of a resolution tree, where the used computation function is left-most (the set of clauses is considered ordered when picking clauses for resolution).

Exercise 187 *Give an SLD resolution of the following clauses:*

C_1 $p(a)$
C_2 $r(x)$
C_3 $q(s(x)) \vee \neg r(f(x)) \vee \neg p(x)$
C_4 $p(s(x)) \vee \neg q(x)$
C_5 $\neg p(s(x))$

where $\{C_1, C_2, C_3, C_4\}$ is the support set, and $\{C_5\}$ is the goal set.

Exercise 188 *Consider the sets of clauses (or the clauses obtained from the formulæ) in other exercises about resolution. Reason about whether an SLD resolution can be given for such sets, in the same circumstances as the original exercises. If it is the case, write the SLD resolution down*

- *considering the set of clauses as ordered*
- *ordering the literals of each clause*
- *giving the complete resolution tree (i.e., running all the paths)*

Exercise 189 (02/2009) *Consider the following set $\mathcal{C}$ of clauses:*

C_1: $\neg t(y)$
C_2: $p(x) \vee t(x) \vee \neg r(x, g(x))$
C_3: $r(h(z), z) \vee \neg p(h(z))$
C_4: $t(y) \vee \neg q(g(y)) \vee p(y)$
C_5: $\neg r(x, y)$
C_6: $q(x) \vee t(x)$

1. *prove by linear input resolution that $\mathcal{C}$ is unsatisfiable;*
2. *does $\{C_2, C_3, C_4, C_5, C_6\}$ guarantee, as a support set, the existence of a directed resolution? Explain why and give a simple example which shows if the set satisfies the required property.*

Solution 29 1. *Linear input resolution (variables are often renamed for clarity, and subscripts usually refer to input clause numbers) :*

$$\begin{array}{lcl} C_1, C_2 & \rightarrow & R_1 : \neg q(g(y_4)) \vee p(y_4) \\ R_1, C_6 & \rightarrow & R_2 : p(y_4) \vee t(g(y_4)) \\ R_2, C_3 & \rightarrow & R_3 : r(h(z_3), z_3) \vee t(g(h(z_3))) \\ R_3, C_1 & \rightarrow & R_4 : r(h(z_3), z_3) \\ R_4, C_5 & \rightarrow & \square \end{array}$$

2. *The correct answer is: $\{C_2, C_3, C_4, C_5, C_6\}$ does guarantee, as a support set, the existence of a directed resolution because it is satisfiable. In fact, directed resolutions on a support set S is guaranteed to be complete provided S is satisfiable.*
In order to prove that $\{C_2, C_3, C_4, C_5, C_6\}$ is satisfiable, it is enough to give a model. As an example for a model, we can simply take the interpretation where t is always true, p is always false and r is always false.

Exercise 190 (02/2009) *Considering the following set of clauses*

C_1: $p(a, x, a)$
C_2: $p(f(x), y, z) \vee \neg p(x, y, z') \vee \neg q(y, z', z)$
C_3: $q(a, x, x)$
C_4: $q(f(y), x, f(z)) \vee \neg q(y, x, z)$

find, by SLD resolution, an answer to the question:

which x makes $p(f(f(a)), f(f(a)), x)$ true?

Solution 30 *Note that this can be directly translated into a logic program. The semantics of this program is arithmetic multiplication: the answer for a generic goal $\neg p(f^m(a), f^n(a), x_0) \vee ans(x_0)$ is $x_0 = f^{mn}(a)$.*

In our case, the initial goal is $G = \neg p(f(f(a)), f(f(a)), x_0) \vee ans(x_0)$, and the answer comes to be $f(f(f(f(a))))$ (representing $2 \times 2 = 4$).

- C_2, G *give* $R_1 =$
  $\neg p(f(a), f(f(a)), z') \vee \neg q(f(f(a)), z', x_0) \vee ans(x_0)$
- C_2, R_1 *give* $R_2 =$
  $\neg p(a, f(f(a)), z_2') \vee \neg q(f(f(a)), z_2', z') \vee \neg q(f(f(a)), z', x_0) \vee ans(x_0)$
- C_1, R_2 *give* $R_3 =$
  $\neg q(f(f(a)), a, z') \vee \neg q(f(f(a)), z', x_0) \vee ans(x_0)$
- C_4, R_3 *give* $R_4 =$
  $\neg q(f(a), a, z_4) \vee \neg q(f(f(a)), f(z_4), x_0) \vee ans(x_0)$
- C_4, R_4 *give* $R_5 =$
  $\neg q(a, a, z_5) \vee \neg q(f(f(a)), f(f(z_5)), x_0) \vee ans(x_0)$
- C_3, R_5 *give* $R_6 =$
  $\neg q(f(f(a)), f(f(a)), x_0) \vee ans(x_0)$
- C_4, R_6 *give* $R_7 =$
  $\neg q(f(a), f(f(a)), z_7) \vee ans(f(z_7))$
- C_4, R_7 *give* $R_8 =$
  $\neg q(a, f(f(a)), z_8) \vee ans(f(f(z_8)))$
- C_3, R_8 *give* $R_9 =$
  $ans(f(f(f(f(a)))))$

Exercise 191 *Given the following clause set:*

C_1: $\neg r(x)$
C_2: $p(y) \vee \neg s(y, h(y)) \vee r(y)$
C_3: $\neg s(z, x)$
C_4: $\neg p(f(g(z))) \vee s(b, h(c)) \vee s(f(g(z)), g(z))$
C_5: $p(x) \vee r(y) \vee \neg q(h(x))$
C_6: $\neg s(x, x) \vee r(a)$
C_7: $s(f(z), z) \vee \neg p(f(z))$
C_8: $s(a, f(c)) \vee p(g(x)) \vee r(z)$

1. *prove that, from the set, it is possible to deduce* $\exists z(\neg q(z) \wedge \neg r(z))$, in two steps*:*
   (a) *identify tautologies (mark with T); clause with pure literals (mark with P); subsumed clauses (mark I, and specify the subsuming clause), and remove them;*
   (b) *apply MGU resolution to the remaining clauses (write the substitutions);*
2. *use known techniques in order to find the value of z which makes the statement at 1 possible;*
3. *is the derivation obtained at 1 (a) ordered? (b) input? (c) directed? Justify.*

Solution 31 *Clause form of the negation of the conclusion:* $C_9 = q(z) \vee r(z)$*, which will be added to the set of clauses.*

1. (a) *no tautological clauses;*
   - C_2 *contains the pure literal* $\neg s(y, h(y))$*;*
   - C_6 *contains the pure literal* $\neg s(x, x)$*;*
   - C_8 *contains the pure literal* $p(g(x))$*;*
   - C_4 *is subsumed by* C_7*:* $C_4 = C_7\alpha \vee s(b, h(c))$*, where* $\alpha = \{z_7/f(z_4)\}$ *(renaming z by* z_4 *or* z_7 *when necessary)*

   therefore, it is possible to remove C_2*,* C_4*,* C_6*,* C_8*;*
   (b) *a possible linear resolution is the following (it is reasonable to start from* C_9 *since it is the conclusion):*

$$\begin{array}{lcl} C_1,\ C_9 & \rightarrow & C_{10} = q(z) & \{\ x_1/z_9\ \} \\ C_{10},\ C_5 & \rightarrow & C_{11} = p(x) \vee r(y) & \{\ z_{10}/h(x_5)\ \} \\ C_{11},\ C_1 & \rightarrow & C_{12} = p(x) & \{\ x_1/y_{11}\ \} \\ C_{12},\ C_7 & \rightarrow & C_{13} = s(f(z), z) & \{\ x_{12}/f(z_7)\ \} \\ C_{13},\ C_3 & \rightarrow & C_{14} = \Box & \{\ z_3/f(z_{13}),\ x_3/z_{13}\ \} \end{array}$$

   where variable have been renamed before and after each step (renaming is for free):
   - *before the step,* C_i *is renamed by* $\{x/x_i, y/y_i, z/z_i\}$*;*
   - *after, all subscripts are removed;*
2. *from the above derivation:* C_9 *becomes* $q(z) \vee r(z) \vee resp(z)$

$$\begin{array}{lcl} C_1,\ C_9 & \rightarrow & C_{10} = q(z) \vee resp(z) & \{\ x_1/z_9\ \} \\ C_{10},\ C_5 & \rightarrow & C_{11} = p(x) \vee r(y) \vee resp(h(x)) & \{\ z_{10}/h(x_5)\ \} \\ C_{11},\ C_1 & \rightarrow & C_{12} = p(x) \vee resp(h(x)) & \{\ x_1/y_{11}\ \} \\ C_{12},\ C_7 & \rightarrow & C_{13} = s(f(z), z) \vee resp(h(f(z))) & \{\ x_{12}/f(z_7)\ \} \\ C_{13},\ C_3 & \rightarrow & C_{14} = resp(h(f(z))) & \{\ z_3/f(z_{13},\ x_3/z_{13}\ \} \end{array}$$

   therefore, the answer is $h(f(z))$ *(for every z); note that, with other derivation, we could have generate the answer* $h(f(g(z)))$*, which is less general and sub-optimal, though correct;*
3. *the answers to this question depend on the derivation*
   - *not ordered, since in the first step* $r(z)$ *is not the first literal;*
   - *input;*
   - *possibly directed, if the support set is* $\{C_1, .., C_8\}$*; anyway, it could also be non-directed if another support set is chosen.*

Exercise 192 (02/2008) *Given the following set of clauses:*

$C_1 : A(x) \vee \neg B(g(x)) \vee C(x)$
$C_2 : \neg C(x)$
$C_3 : A(x) \vee C(x) \vee \neg D(x, g(x))$
$C_4 : C(x) \vee \neg D(x, y)$
$C_5 : B(x) \vee C(x)$
$C_6 : \neg A(f(x)) \vee D(f(x), x)$

(a) Demonstrate, using resolution, that it is unsatisfiable. (b) Is the previous refutation linear? Is it input? What would be necessary for it to be a directed refutation? Briefly motivate your answer.

Exercise 193 (02/2006) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : \neg P(x) \vee Q(x) \vee \neg R(x,y) \\ C_2 : \neg D(x) \vee \neg Q(y) \vee \neg R(x,y) \\ C_3 : \neg D(x) \vee P(x) \\ C_4 : D(f(x)) \\ C_5 : D(a) \\ C_6 : R(x, f(x)) \end{array}$$

(a) Demonstrate, using resolution, that it is unsatisfiable. (b) Is the previous refutation linear? Is it input? What would be necessary for it to be a directed refutation? Briefly motivate your answer.

Exercise 194 (06/2006) *Given the following set of clauses:*

$$\begin{array}{l} C_1 : N(x,y,z) \vee \neg N(u,y,v) \vee \neg N(x,w,u) \vee \neg N(w,z,v) \\ C_2 : N(x,y,z) \vee \neg N(u,y,v) \vee \neg N(u,x,w) \vee \neg N(w,v,z) \\ C_3 : N(a,x,x) \\ C_4 : N(x,a,f(x)) \\ C_5 : N(f(x),a,x) \\ C_6 : \neg N(b,a,b) \end{array}$$

1. *define a support set such that the guided resolution strategy is complete for this set of clauses and justify why it is;*
2. *using directed resolution, decide if the set of clauses is unsatisfiable or not, and justify your conclusion.*

Exercise 195 (02/2007) *Given the following set of clauses*

$$\begin{array}{l} C_1 : \neg C(x) \vee D(x) \vee B(x, g(x)) \\ C_2 : \neg C(x) \vee D(x) \vee E(g(x)) \\ C_3 : \neg B(a,y) \vee A(y) \\ C_4 : \neg A(x) \vee \neg E(x) \\ C_5 : \neg A(f(y)) \vee \neg D(y) \\ C_6 : C(a) \end{array}$$

Prove, using resolution, that the deductive structure

$$[C_1, C_2, C_3, C_4, C_5, C_6] \vdash \exists x \neg A(f(x)) \wedge \exists y \neg A(y)$$

is correct. Briefly motivate your answer.

Exercise 196 (09/2008) *Demonstrate, by using resolution, that the following set of clauses is unsatisfiable:*

$$\begin{array}{l} C_1 : \neg P(f(b)) \vee \neg P(y) \vee Q(y) \\ C_2 : R(z, g(z)) \\ C_3 : \neg R(a,x) \vee \neg Q(x) \\ C_4 : P(u) \end{array}$$

Exercise 197 (07/2009) *Mark with an X the correct statements, and fix the incorrect ones:*

1. *a formula is unsatisfiable iff there exists a set of ground instances of its clauses which is unsatisfiable;*
2. *a formula is unsatisfiable iff it has no countermodels;*
3. *a set of clauses is unsatisfiable iff it is possible to deduce the empty clause from it by means of input linear resolution;*
4. *an Herbrand model is enough in order to prove the satisfiability of a fórmula;*
5. *a formula is unsatisfiable iff its associated semantic tree is finite whenever the successors of failure nodes are not considered.*

Solution 32 – *"set" → "finite set"*
- *"countermodels" → "models"*
- *remove "input"*
- *OK*
- *"its associated semantic tree is finite" → "its associated semantic tree is closed and finite"*

Exercise 198 (07/2009) *Prove, by MGU resolution, that* $\forall z\exists x\exists y(A(f(x),y,h(z,b)) \wedge B(g(y,f(x))))$ *can be deduced from:*

$$\begin{array}{l} C_1 : A(x,y,h(y,z)) \vee \neg C(z,x) \vee \neg D(f(z),u) \\ C_2 : C(b,x) \vee \neg D(x,y) \\ C_3 : D(x,x) \\ C_4 : B(g(x,f(x)) \\ C_5 : B(g(x,f(y)) \\ C_6 : \neg D(x,f(x)) \end{array}$$

Is this derivation liner? Is it directed?

Exercise 199 (06/2010) *1. Prove that* $\forall x\exists y(p(x,y) \rightarrow \exists zt(y,g(g(z))))$ *can be deduced from the following clause set* C*:*

$$\begin{array}{l} C_1 : \neg p(x,y) \vee q(y,x) \vee t(x,g(y)) \\ C_2 : r(y) \vee t(g(x),c) \vee \neg p(g(x),y) \\ C_3 : r(y) \vee \neg p(x,y) \\ C_4 : \neg s(y) \vee \neg q(g(a),y) \\ C_5 : \neg r(g(y)) \vee s(y) \\ C_6 : \neg t(z,g(c)) \vee \neg p(x,c) \end{array}$$

In order to prove the result

- *apply the rule of pure literal elimination*
- *apply UMG resolution to the result of the previous step, specifying at each step the computed MGU*

2. *Is this derivation linear? Is it input? Motivate.*
3. *If* $\{C_2, C_3, C_4\}$ *were chosen as the support set, would the derivation be directed? Motivatecomo conjunto soporte, la derivación ¿sería dirigida? Justificar la respuesta.*
4. *does the same support set* $\{C_2, C_3, C_4\}$ *satisfy the completeness condition for the directed resolution? Motivate.*

Exercise 200 (06/2010) *Factorize the following clause:*

$$q(x,f(g(x))) \vee \neg r(y,x) \vee q(y,f(g(h(x)))) \vee q(w,f(g(h(z))))$$