CHAPTER III: CONICS AND QUADRICS

4. QUADRICS

Let $\mathbb{P}_3 = \mathbb{P}(\mathbb{R}^4)$ be the real projective tridimensional space.

Definition. A quadric $\overline{Q}$ in $\mathbb{P}_3$ determined by a quadratic form $\omega \colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ is the set of points of $\mathbb{P}_3$ defined by:

$$\overline{Q} = \{X \in \mathbb{P}_3 \mid \omega(X) = 0\}$$

Let $\mathcal{R} = \{O, B\}$ be a coordinate system in $\mathbb{A}_3$ and let

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}$$

be the matrix associated to the quadratic form ω then

$$\begin{aligned} \overline{Q} &= \{X \in \mathbb{P}_3 \mid X^t A X = 0\} \\ &= \left\{ [x_0, x_1, x_2, x_3] \in \mathbb{P}_3 \mid \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x_i x_j = 0 \right\} \end{aligned}$$

The affine quadric defined by the quadratic form ω is the subset Q of $\mathbb{A}_3$ defined by

$$Q = \{X \in \mathbb{A}_3 \mid \omega(\tilde{X}) = 0\},$$

where $\tilde{X} = (1, x_1, x_2, x_3)$, with $(x_1, x_2, x_3) \in \mathbb{A}_3$. It is verified that $Q \subset \overline{Q}$.

4.1 Singular points and projective classification

Let $\overline{Q}$ be a projective quadric determined by a quadratic form $\omega\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$, with polar form $f\colon \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ and associated matrix A with respect to certain coordinate system.

Definitions.

- We say that two points $A, B \in \mathbb{P}_3$ are *conjugated* with respect to $\overline{Q}$ if $f(A, B) = 0$.
- We say that a point $P \in \mathbb{P}_3$ is an *autoconjugated point* with respect to $\overline{Q}$ if $\omega(P) = f(P, P) = 0$.
- We say that a point $P \in \mathbb{P}_3$ is a *singular point* of $\overline{Q}$ if it is conjugated with every point of $\mathbb{P}_3$; this is, $f(P, X) = 0$ for every point $X \in \mathbb{P}_3$. This is, if

$$f(P, X) = P^T AX = 0, \ \forall X \in \mathbb{P}_3,$$

or equivalently,

$$P^T A = 0.$$

- We say that a point $P \in \mathbb{P}_3$ is a *regular point* of $\overline{Q}$ if it is not a singular point
- The quadric $\overline{Q}$ is *non degenerate, regular or ordinary* if it does not have singular points.
- The quadric $\overline{Q}$ is *degenerate or singular* if it has a singular point.

Observations: Let $\overline{Q}$ be a projective quadric generated by a quadratic form ω, with polar form f and associated matrix A.

1. Let $\mathrm{sign}(\overline{Q})$ be the set of singular points of $\overline{Q}$; this is,

$$\begin{aligned}\mathrm{sign}(\overline{Q}) &= \{X \in \mathbb{P}_3 \mid f(X,Y) = 0, \text{ for every } Y \in \mathbb{P}_3\} \\ &= \{X \in \mathbb{P}_3 \mid AX = 0\}.\end{aligned}$$

   We have

$$\dim(\mathrm{sign}(\overline{Q})) = 3 - \mathrm{rank}(A).$$

2. If $X \in \mathbb{P}_3$ is a singular point, then $X \in \overline{Q}$.

   Proof. We have to check that $\omega(X) = 0$. We have $\omega(X) = f(X,X) = 0$ as X is conjugated with any point, in particular with itself.

3. The line determined by a singular point X and any other point of the quadric, $Y \in \overline{Q}$, is contained on the quadric.

   Proof. As X is singular we know that $\omega(X) = 0$ and $f(X,Y) = 0$ and as Y belongs to the quadric $\omega(Y) = 0$. Any point of the line determined by X and Y has the form $Z = \lambda X + \mu Y$. We have to check whether $\omega(Z) = 0$. We have:

$$
\begin{aligned}
\omega(Z) &= \omega(\lambda X + \mu Y) = f(\lambda X + \mu Y, \lambda X + \mu Y) \\
&= f(\lambda X, \lambda X + \mu Y) + f(\mu Y, \lambda X + \mu Y) \\
&= f(\lambda X, \lambda X) + f(\lambda X, \mu Y) + f(\mu Y, \lambda X) + f(\mu Y, \mu Y) \\
&= \lambda^2 f(X,X) + 2\lambda\mu f(X,Y) + \mu^2 f(Y,Y) \\
&= \lambda^2 \underbrace{\omega(X)}_{0} + 2\lambda\mu \underbrace{f(X,Y)}_{0} + \mu^2 \underbrace{\omega(Y)}_{0} = 0.
\end{aligned}
$$

4. All the points that belong to the line determined by two singular points are singular.

Proof. Let $Z = \lambda X + \mu Y$ be any point of the line formed by two singular points X and Y. We have to check that $f(Z,T) = 0$, for every $T \in \mathbb{P}_3$. We have:

$$\begin{aligned} f(Z,T) &= f(\lambda X + \mu Y, T) \\ &= f(\lambda X, T) + f(\mu Y, T) \\ &= \lambda \underbrace{f(X,T)}_{0} + \mu \underbrace{f(Y,T)}_{0} = 0. \end{aligned}$$

5. If the quadric $\overline{Q}$ contains a singular point, then $\overline{Q}$ is formed by lines that contain that point.

4.1.1 Projective classification

1. If $\det A \neq 0$, then the quadric $\overline{Q}$ is *ordinary or not degenerate*.
2. If $\det A = 0$, then the quadric $\overline{Q}$ is *degenerate*.

   a) If $\text{rank}(A) = 3$, then $\overline{Q}$ has an unique singular point P.
   - If P is a proper point, then $\overline{Q}$ is a *cone* with vertex P.
   - If P is an improper point, then $\overline{Q}$ is a *cylinder*.

   b) If $\text{rank}(A) = 2$, then $\overline{Q}$ has a line of singular points and $\overline{Q}$ is a *pair of planes* with intersection the line of singular points.

   c) If $\text{rank}(A) = 1$, then $\overline{Q}$ has a plane of singular points and $\overline{Q}$ is a *double plane*.

4.2 Polarity defined by a quadric

Let $\overline{Q}$ be a quadric with polar form f and associated matrix A. Let us consider $P \in \mathbb{P}_3$, we call polar variety of P with respect to the quadric $\overline{Q}$ to the set of points in $\mathbb{P}_3$ conjugated with P; this is,

$$\begin{aligned} V_P &= \{X \in \mathbb{P}_3 \mid f(P, X) = 0\} \\ &= \{X \in \mathbb{P}_3 \mid P^t A X = 0\}. \end{aligned}$$

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$$\begin{aligned} V_P &= \{X \in \mathbb{P}_3 \mid f(P, X) = 0\} \\ &= \{X \in \mathbb{P}_3 \mid P^t A X = 0\}. \end{aligned}$$

If $P \in \mathbb{P}_3$ is a singular points, then $V_P = \mathbb{P}_3$.

If $P \in \mathbb{P}_3$ is not a singular point, then V_P is the tangent plane π_P and we call it polar plane of P with respect to the quadric $\overline{Q}$:

$$\pi_P = \{X \in \mathbb{P}_3 \mid P^t A X = 0\}.$$

Definition. Given a plane π of the space $\mathbb{P}_3$, we call **pole** of the plane π with respect to the quadric $\overline{Q}$ to the point whose polar plane is π; this is, $\pi_P = \pi$.

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If the equation of the plane π is

$$\pi \equiv u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = U^TX = 0,$$

$$\text{with } U = (u_0, u_1, u_2, u_3) \text{ and } X = (x_0, x_1, x_2, x_3),$$

then $\pi_P = \pi$ if and only if

$$P^TAX = U^TX, \text{ for every } X \in \mathbb{P}_3$$

equivalently,

$$P^TA = U^T \Longleftrightarrow AP = U.$$

And if the quadric $\overline{Q}$ is not degenerate (therefore, $\det A \neq 0$), then $P = A^{-1}U$.

Theorem. If the point P belongs to the polar plane of a point R, then the point R is in the polar plane of P.

This is due to the condition of conjugation $f(P, R) = 0$; it is symmetric in P and R.

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As we have seen, given a quadric $\overline{Q}$, every non sigular point P is assigned a plane (its polar plane) and reciprocally, each plane π is assigned a point (its pole).

Definition. We call polarity defined by a quadric $\overline{Q}$ to the transformation sending each non singular point of $\overline{Q}$ to its polar plane. This is,

$$\mathbb{P}_3 \setminus \mathrm{sign}(\overline{Q}) \longrightarrow \text{Planes of } \mathbb{P}_3$$
$$P \longmapsto \pi_P$$

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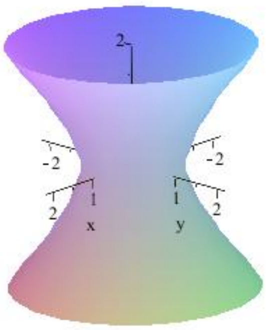
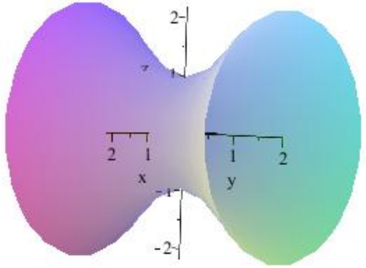
Fundamental theorem of polarity The polar planes of the points of a plane π of $\mathbb{P}_3$, with respect to a regular quadric $\overline{Q}$, contain the pole of π.

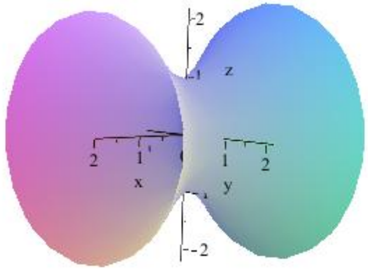
INTERSECTION WITH COORDINATE AXES:



$$x^2 + y^2 - z^2 = 1$$



$$x^2 - y^2 + z^2 = 1$$



$$-x^2 + y^2 + z^2 = 1$$

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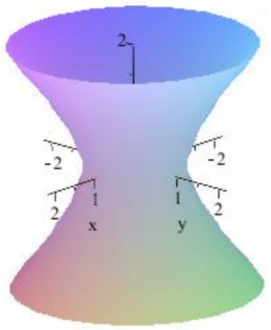

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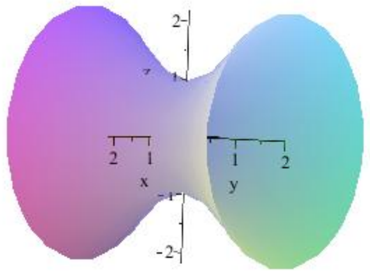

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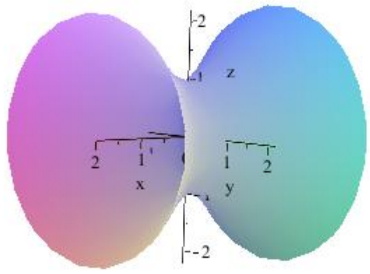

$$-x^2 + y^2 + z^2 = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \begin{cases} z \text{ axis} & \text{No real intersection} \\ y \text{ axis} & (0, \pm b, 0) \\ x \text{ axis} & (\pm a, 0, 0) \end{cases}$$

4.3 Intersection of a quadric with a line

Let $\overline{Q}$ be a projective quadric with polar form f and associated matrix A. Let r be the projective line which contains the independent points $P = [(p_0, p_1, p_2, p_3)]$ and $Q = [(q_0, q_1, q_2, q_3)]$.

A point $X \in \mathbb{P}_3$ is in the intersection between the conic and the line if and only if:

$$\begin{cases} X \in r \\ X \in \overline{Q} \end{cases} \Longleftrightarrow \begin{cases} X = \lambda P + \mu Q \\ \omega(X) = 0 \end{cases} \Longleftrightarrow \begin{cases} X = \lambda P + \mu Q \\ \omega(\lambda P + \mu Q) = 0 \end{cases}$$

The condition $\omega(\lambda P + \mu Q) = 0$ is written:

$$0 = \lambda^2 \omega(P) + 2\lambda\mu f(P, Q) + \mu^2 \omega(Q).$$

Dividing the former equation by μ^2 and writing $t = \lambda/\mu$ we obtain the following second degree equation:

$$0 = \omega(P)t^2 + 2f(P, Q)t + \omega(Q)$$

with discriminant

$$\Delta = f(P,Q)^2 - \omega(P)\omega(Q).$$

- If $f(P,Q) = 0$, $\omega(P) = 0$ and $\omega(Q) = 0$, then $P, Q \in \overline{Q}$ and, therefore, $r \subset \overline{Q}$.
- If not all the coefficients of the second degree equation $0 = \omega(P)t^2 + 2f(P,Q)t + \omega(Q)$ are zero, then there are two intersection points (the two solutions of the equation).

1. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) > 0$, the line, and the quadric intersect in two different real points. The line is called secant line to the quadric.
2. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) = 0$, the line and the quadric intersect in a double point. The line is called tangent line to the quadric.
3. If $\Delta = f(P,Q)^2 - \omega(P)\omega(Q) < 0$, the line and the conic intersect in two different improper points. The line is called exterior line to the quadric.

4.3.1 Tangent variety to a quadric

Definition. The tangent variety to a quadric $\overline{Q}$ in a point $P \in \mathbb{P}_3$, is the set of points $X \in \mathbb{P}_3$ such that the line that joins P and X is tangent to the quadric $\overline{Q}$; this is,

$$\begin{aligned} T_P\overline{Q} &= \{X \in \mathbb{P}_3 \mid \text{line } XP \text{ is tangent to } \overline{Q}\} \\ &= \{X \in \mathbb{P}_3 \mid \Delta = f(P,X)^2 - \omega(P)\omega(X) = 0\} \\ &= \{X \in \mathbb{P}_3 \mid f(P,X)^2 = \omega(P)\omega(X)\}. \end{aligned}$$

1. $T_P\overline{Q}$ is a degenerate quadric which has P as singular point.
2. If $P \in \overline{Q}$ is a regular point, then

$$\begin{aligned} T_P\overline{Q} &= \{X \in \mathbb{P}_3 \mid f(P,X)^2 = 0\} \\ &= \{X \in \mathbb{P}_3 \mid P^t A X = 0\} \end{aligned}$$

   is a plane, called the tangent plane to $\overline{Q}$ on P. In fact, it is the polar plane of the point P; this is, $T_P\overline{Q} = \pi_p$.
3. If $P \in \overline{Q}$ is a singular point, then $T_P\overline{Q} = \mathbb{P}_3$.

4.4 Affine classification and notable elements of quadrics

Let $\overline{\mathbb{A}}_3 = \mathbb{P}(\mathbb{R}^4)$ be the projectivized affine space, with coordinate system $\mathcal{R} = \{O, B\}$. And let ω be a quadratic form with associated matrix A. Let

$$\overline{Q} = \{X \in \mathbb{P}_3(\mathbb{R}^4) \mid \omega(X) = 0$$

be a projective quadric with affine quadric

$$Q = \overline{Q} \cap \mathbb{A}_3 = \{X \in \mathbb{A}_3 \mid \omega(\tilde{X}) = 0\}, \text{ where } \tilde{X} = (1, x_1, x_2, x_3).$$

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$$Q = \overline{Q} \cap \mathbb{A}_3 = \{X \in \mathbb{A}_3 \mid \omega(\tilde{X}) = 0\}, \text{ where } \tilde{X} = (1, x_1, x_2, x_3).$$

Definition. We call center of an affine quadric Q to the pole of the plane at infinity, if it exists. If that point is contained in the plane at infinity then the quadric has an improper center, otherwise a proper center.

The pole of the plane at infinity is the point P such that $P^t A = (1, 0, 0, 0)$.

Proposition. The proper center of an affine quadric is its center of symmetry. Any line that contains the center intersects the quadric in two symmetric points with respect to the center.

4.4.2 Relative position of the quadric and the plane at infinity

Let $\pi_\infty \equiv x_0 = 0$ be the equation of the plane at infinity and let us consider the projective quadric $\overline{Q}$ determined by a quadratic form ω with associated matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

We have:

$$\overline{Q} \cap \pi_\infty = \{X \in \pi_\infty \mid \omega(X) = 0\} = \left\{(0, x_1, x_2, x_3) \mid X^t A X = 0\right\}$$

this is,

$$\overline{Q} \cap \pi_\infty \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0,$$

then $\overline{Q} \cap \pi_\infty$ is a conic in the plane at infinity π_∞ with matrix

$$A_{00} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Proposition. The quadric $\overline{Q}$ has a proper center if and only if $\det A_{00} \neq 0$. Besides,

- If $\det A_{00} \neq 0$, then the conic $\overline{Q} \cap \pi_\infty$ is regular and $\overline{Q}$ has a proper center.
- If $\det A_{00} = 0$, then the conic $\overline{Q} \cap \pi_\infty$ is degenerate and $\overline{Q}$ has no proper center.

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Quadrics with proper center are: ellipsoids, hyperboloids and cones.

Definition.

- We call diameter of a quadric $\overline{Q}$ to every line that contains the center of $\overline{Q}$.
- We call diametral plane of a quadric $\overline{Q}$ to the planes that contain the center of $\overline{Q}$.
- Two diameters D and D' are said conjugated if their improper points are conjugated.
- We call diametral polar plane of a diameter D to the polar plane of its improper point.

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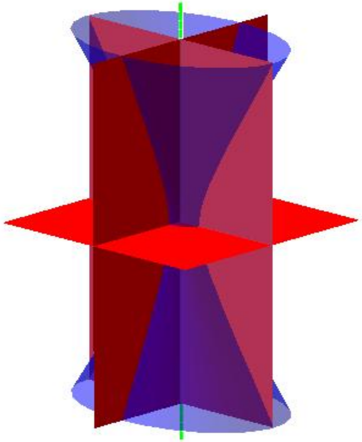
Axes of a quadric with proper center

Definition. We call axis of a quadric $\overline{Q}$ to a diameter which is perpendicular to its diametral polar plane, which is called main plane of the quadric. A vertex of $\overline{Q}$ is the intersection point of $\overline{Q}$ with an axis of $\overline{Q}$.

MAIN PLANES AND CENTER:

$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{25}=1$$



Planes of symmetry

$x=0, y=0, z=0$

Center of symmetry

$(0,0,0)$

$\frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{25}=1$, the z axis is an axis of revolution.

$\overline{Q}$ has proper center Z, A_{00} is nonsingular, so its eigenvalues λ_1, λ_2 and λ_3 are nonzero. Let v_1, v_2 and v_3 be eigenvectors associated to λ_1, λ_2 and λ_3 respectively (we choose them to be orthogonal two by two).

$\mathcal{R} = \{Z, \{v_1, v_2, v_3\}\}$ is an orthogonal coordinate system.

The lines $E_1 = Z + \langle v_1 \rangle$, $E_2 = Z + \langle v_2 \rangle$ $E_3 = Z + \langle v_3 \rangle$ are axes of $\overline{Q}$ but there may be more:

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1. If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then $\overline{Q}$ has only three axes E_1, E_2, E_3.
2. An eigenvalue is double $\lambda_1 = \lambda_2 \neq \lambda$ and the other is simple. The dimensions of the eigenspaces are $\dim\langle v_1, v_2 \rangle = 2$ and $\dim\langle v_3 \rangle = 1$. Then $Z + V_1$ is a plane of axes perpendicular to the axis $Z + V_3$. $\overline{Q}$ is a revolution quadric with axis of revolution $Z + V_3$.
3. One triple eigenvalue $\lambda_1 = \lambda_2 = \lambda_3$. Every diameter is an axis and the quadric is a sphere.

4.4.5 Asymptotic cones

Definition. We call asymptotes of a quadric $\overline{Q}$ to the tangent lines to its conic of improper points.

Definition. Let $\overline{Q}$ be a projective quadric with proper center Z. The tangent variety to the quadric $\overline{Q}$ from its center $Z\,[z_0, z_1, z_2, z_3]$ is a cone called asymptotic cone.

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The equation of the asymptotic cone is obtained as follows:

$$f(Z,X)^2 - \omega(Z)\omega(X) = 0 \Longleftrightarrow (Z^tAX)(Z^tAX) - (Z^tAZ)(X^tAX) = 0$$

$$\Longleftrightarrow x_0^2 - z_0(X^tAX) = 0 \Longleftrightarrow x_0^2 - \frac{\det A_{00}}{\det A}(X^tAX) = 0$$

equivalently

$$\frac{\det A}{\det A_{00}} x_0^2 - \overline{Q} = 0.$$

The quadrics of elliptic type have an imaginary asymptotic cone and the quadrics of hyperbolic type have a real asymptotic cone.

A generatrix of the cone (a line of the cone) are the diameters tangent to the quadric.

We call asymptotic plane to a polar plane of the points of the improper conic $\overline{Q} \cap \pi_\infty = C$ (if there exists any).

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Example

Let us consider the quadric $Q \equiv x_1^2 + 3x_3^2 + 4x_1x_2 + 2x_3 + 2 = 0$. The matrix of Q is:

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

The determinant of A is $\det A = -20$, thus it is a quadric with proper center Z.

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \rho \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{cases} 2z_0 + z_3 = \rho \\ z_1 + 2z_2 = 0 \\ 2z_1 = 0 \\ z_0 + 3z_3 = 0 \end{cases}$$

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \rho \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{cases} 2z_0 + z_3 = \rho \\ z_1 + 2z_2 = 0 \\ 2z_1 = 0 \\ z_0 + 3z_3 = 0 \end{cases}$$

The center is $Z = [1, 0, 0, -1/3]$ and the equation of the asymptotic cone is

$$\frac{\det A}{\det A_{00}} x_0^2 - \overline{Q} = 0 \Longleftrightarrow \frac{20}{12} x_0^2 - (x_1^2 + 3x_3^2 + 4x_1x_2 + 2x_3x_0 + 2x_0^2) = 0$$

$$\Longleftrightarrow 2x_0x_3 + 4x_1x_2 + \frac{1}{3}x_0^2 + x_1^2 + 3x_3^2 = 0.$$

4.5 Metric invariants of a quadric $\overline{Q}$

Let us consider the quadric $\overline{Q}$ with associated matrix A; this is, $\overline{Q} \equiv X^T A X = 0$. The following values are euclidean invariants of the quadric:

- $\det A$
- Eigenvalues of A_{00}: $\lambda_1, \lambda_2, \lambda_3$ or equivalently:

$$\det A_{00},\ \operatorname{tr} A_{00} = a_{11} + a_{22} + a_{33},\ J = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$$

where

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix} \text{ and } A_{00} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

The following identities are satisfied:

- $\det A_{00} = \lambda_1\lambda_2\lambda_3$
- $J = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$
- $\operatorname{tr} A_{00} = \lambda_1 + \lambda_2 + \lambda_3$

The charasteristic equation of A_{00} is:

$$|A_{00} - \lambda I_3| = -\lambda^3 + \operatorname{tr} A_{00}\lambda^2 - J\lambda + \det A_{00} = 0.$$

Therefore, λ_1, λ_2 and λ_3 are the roots of the equation $|A_{00} - \lambda I_3| = 0$.

If $\det A_{00} \neq 0$, then the conic $\overline{Q} \cap \pi_\infty$ is regular and $\overline{Q}$ has a center.

If $\det A_{00} = 0$, then the conic $\overline{Q} \cap \pi_\infty$ is not regular. It is a quadric of parabolloid type, it may not have a center, have a line of centers or even have a plane of centers.

4.5.1 Classification of quadrics with PROPER CENTER, $\det A_{00} \neq 0$. There exists a coordinate system in which the matrix of the quadric is

$$\begin{pmatrix} d_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},$$

with $\det A_{00} = \lambda_1\lambda_2\lambda_3 \neq 0$, therefore the reduced equation of the affine quadric ($x_0 = 1$) is

$$d_0 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

with $d_0 = \frac{\det A}{\det A_{00}}$.

If $\det A = d_0\lambda_1\lambda_2\lambda_3 \neq 0$ (this is, $\operatorname{rank}(A) = 4$) then the quadric is ordinary, it has no singular points. We can distinguish two cases:

1. the eigenvalues of A_{00} have the same sign
2. two of the eigenvalues of A_{00} have the same sign and the third the opposite sign.

1. If $\text{sign}(\lambda_1) = \text{sign}(\lambda_2) = \text{sign}(\lambda_3)$ ($+ + +$ o $- - -$), we say that A_{00} has signature 3, $\text{sig}\, A_{00} = 3$, and we can encounter the following cases:

   a) If $\text{sign}(d_0) = \text{sign}(\lambda_1) = \text{sign}(\lambda_2) = \text{sign}(\lambda_3)$, then $\det A > 0$ and the reduced equation of the affine quadric is

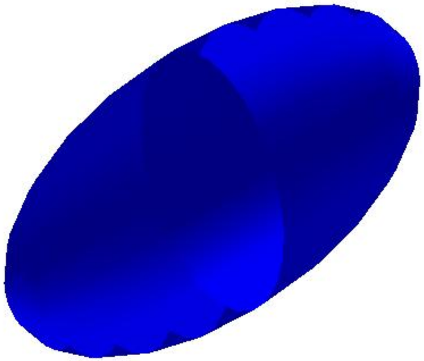
$$1 = -\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}$$

   where $a^2 = d_0/\lambda_1$, $b^2 = d_0/\lambda_2$ and $c^2 = d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *imaginary ellipsoid*.

   b) If $\text{sign}(d_0) \neq \text{sign}(\lambda_1) = \text{sign}(\lambda_2) = \text{sign}(\lambda_3)$, then $\det A < 0$ and the reduced equation of the affine quadric is

$$1 = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}$$

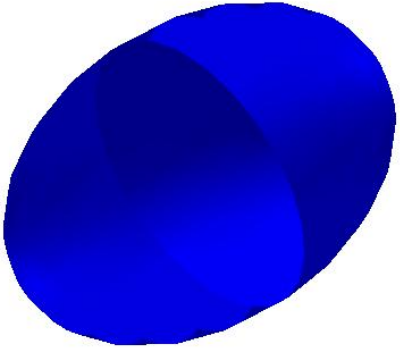
   where $a^2 = -d_0/\lambda_1$, $b^2 = -d_0/\lambda_2$ and $c^2 = -d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *ellipsoid*, and if besides $a^2 = b^2 = c^2$ we obtain a *sphere*.



$$\frac{x^2}{25}+\frac{y^2}{4}+\frac{z^2}{9}=1$$

$\downarrow$

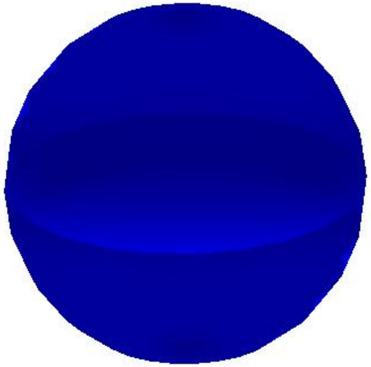
Ellipsoid



$$\frac{x^2}{9}+\frac{y^2}{4}+\frac{z^2}{4}=1$$

$\downarrow$

Revolution Ellipsoid



$$\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{4}=1$$

$\downarrow$

Esfere

2. If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) \neq \mathrm{sign}(\lambda_3)$ ($+ + -$ o $- - +$) we say that A_{00} has *signature* 1, $\mathrm{sig}\, A_{00} = 1$, and we can encounter the following cases:

   a) If $\mathrm{sign}(d_0) \neq \mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) \neq \mathrm{sign}(\lambda_3)$, then $\det A > 0$ and the reduced equation of the affine quadric is

$$1 = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2}$$

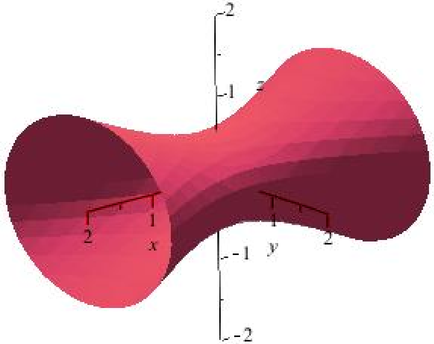
   where $a^2 = -d_0/\lambda_1$, $b^2 = -d_0/\lambda_2$ and $c^2 = d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *hyperbolic hyperboloid*.

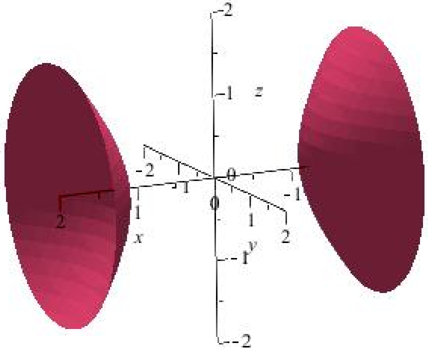
   b) If $\mathrm{sign}(d_0) = \mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) \neq \mathrm{sign}(\lambda_3)$, then $\det A < 0$ and the reduced equation of the quadric is

$$1 = -\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}$$

   where $a^2 = d_0/\lambda_1$, $b^2 = d_0/\lambda_2$ and $c^2 = -d_0/\lambda_3$ (as the three of them are positive) which is the equation of an *elliptic hyperboloid*.

RULED $-4x^2 + 9y^2 + 16z^2 = 5$ NON RULED $-2x^2 + 2y^2 + 2z^2 = -3$





$$A = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}, \det(A) > 0$$

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \det(A) < 0$$

If $\det A = d_0\lambda_1\lambda_2\lambda_3 = 0$ (this is, $d_0 = 0$ and $\mathrm{rank}(A) = 3$) then they are degenerate quadrics with reduced equation:

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

We can distinguish two cases:

1. If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) = \mathrm{sign}(\lambda_3)$, the reduced equation of the affine quadric is

   $$0 = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$$

   which is the equation of an *imaginary cone*.

2. If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) \neq \mathrm{sign}(\lambda_3)$, the reduced equation of the affine quadric is of the form

   $$0 = a^2 x_1^2 + b^2 x_2^2 - c^2 x_3^2$$

   which is the equation of an *cone*.

Table of classification of quadrics with proper center

$\det A_{00} \neq 0$

- $\text{rank}(A) = 4$ Regular
  - $\text{sig}\, A_{00} = 3$ Ellipsoids
    - $\det A > 0$ imaginary
    - $\det A < 0$ real
  - $\text{sig}\, A_{00} = 1$ Hyperboloids
    - $\det A > 0$ hyperbolic
    - $\det A < 0$ elliptic
- $\text{rank}(A) = 3$ Cones
  - $\text{sig}\, A_{00} = 3$ Imaginary cone with a real point
  - $\text{sig}\, A_{00} = 1$ Real cone

4.5.2 Classification of the quadrics with IMPROPER CENTER, $\det A_{00} = 0$.

Since $\det A_{00} = \lambda_1\lambda_2\lambda_3 = 0$ we can assume $\lambda_3 = 0$ and hence $J = \lambda_1\lambda_2$.
In certain coordinate system the matrix of the quadric is

$$\begin{pmatrix} d & 0 & 0 & b_{03} \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ b_{03} & 0 & 0 & 0 \end{pmatrix}$$

with $\det A = -b_{03}^2\lambda_1\lambda_2$.

4.5.2 Classification of the quadrics with IMPROPER CENTER, $\det A_{00} = 0$.

Since $\det A_{00} = \lambda_1\lambda_2\lambda_3 = 0$ we can assume $\lambda_3 = 0$ and hence $J = \lambda_1\lambda_2$.
In certain coordinate system the matrix of the quadric is

$$\begin{pmatrix} d & 0 & 0 & b_{03} \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ b_{03} & 0 & 0 & 0 \end{pmatrix}$$

with $\det A = -b_{03}^2\lambda_1\lambda_2$.

Thus the reduced equation of the quadric is:

$$\lambda_1 x^2 + \lambda_2 y^2 + b_{03} z + d = 0.$$

If $J = \lambda_1\lambda_2 \neq 0$ we can distinguish various cases:

1. If $\det A \neq 0$ (this is, $b_{03} \neq 0$):

   a) If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2)$, this is $J > 0$, the reduced equation of the affine quadric is of the form

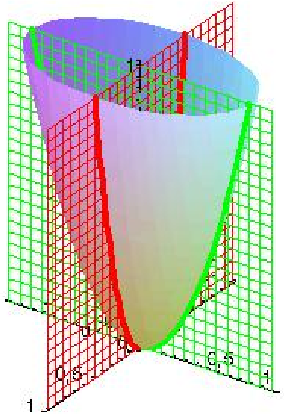
   $$0 = dx_3 + a^2x_1^2 + b^2x_2^2$$

   which is the equation of an *elliptic paraboloid*.

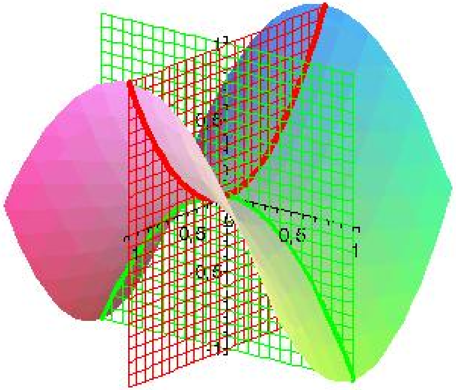
   b) If $\mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$, this is $J < 0$, the reduced equation of the affine quadric is of the form

   $$0 = dx_3 + a^2x_1^2 - b^2x_2^2$$

   which is the equation of an *hyperbolic paraboloid*.



$$z = x^2 + y^2$$
$$\downarrow$$
Elliptic Paraboloid



$$z = x^2 - y^2$$
$$\downarrow$$
Hyperbolic Paraboloid

2. If $\det A = 0$ (this is, $b_{03} = 0$) the reduced equation of the affine quadric is $0 = d + \lambda_1 x_1^2 + \lambda_2 x_2^2$.

   a) If $d \neq 0$ we have

      1) If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2)$, this is $J > 0$, the reduced equation of the affine quadric is of the form

$$0 = d + a^2 x_1^2 + b^2 x_2^2$$

      which is the equation of an *elliptic imaginary cylinder* if $d > 0$ or *elliptic cylinder* if $d < 0$.

      2) If $\mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$, this is $J < 0$, the reduced equation of the affine quadric is of the form

$$0 = d + a^2 x_1^2 - b^2 x_2^2$$

      which is the equation of a *hyperbolic cylinder*.

b) If $b_{00} = 0$ the reduced equation of the quadric is

$$0 = \lambda_1 x_1^2 + \lambda_2 x_2^2.$$

1) If $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2)$, this is $J > 0$, the affine quadric is a *pair of imaginary planes which intersect in a line*.
2) If $\mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$, this is $J < 0$, the affine quadric is a *pair of planes which intersect in a line*.

If two of the eigenvalues of A_{00} vanish (suppose $\lambda_2 = \lambda_3 = 0$), hence: $\det A = 0$, $\det A_{00} = 0$, $J = 0$ and $\operatorname{tr} A_{00} = \lambda_1$.

In certain coordinate system the matrix of the quadric is

$$\begin{pmatrix} b_{00} & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the reduced equation of the quadric is

$$0 = b_{00} + \lambda_1 x_1^2.$$

1. If $b_{00} \neq 0$ we have

   a) If $\operatorname{sign}(b_{00}) = \operatorname{sign}(\lambda_1)$, the reduced equation of the affine quadric is of the form

   $$0 = p^2 + a^2 x_1^2$$

   and the affine quadric is a *pair of imaginary parallel planes* .

b) If $\mathrm{sign}(b_{00}) \neq \mathrm{sign}(\lambda_1)$, the reduced equation is of the form

$$0 = p^2 - a^2 x_1^2 = (p + a x_1)(p - a x_1)$$

and the affine quadric is a *pair of parallel planes*.

Table of classification of quadrics with $\det A_{00} = 0$

$\det A_{00} = 0$

- $\mathrm{rank}(A) = 4$ Regular
  - $J > 0$ Elliptic paraboloid
  - $J < 0$ Hyperbolic paraboloid
- $\mathrm{rank}(A) = 3$ Cylinders
  - $J > 0$ Real elliptic cylinder
  - $J < 0$ Hyperbolic cylinder
  - $J = 0$ Parabolic cylinder
- $\mathrm{rank}(A) = 2$ Pair of planes
  - $J > 0$ Pair of imaginary planes (line)
  - $J < 0$ Pair of secant planes
  - $J = 0$ Pair of parallel planes
    - imaginary
    - real
- $\mathrm{rank}(A) = 1$ double plane