CHAPTER I: LINEAR ALGEBRA

1. MATRICES

Let A be a matrix with m rows and n columns, we will say that A has size $m \times n$.

We will write $A = (a_{ij})$ with $i = 1, \ldots, m$, $j = 1, \ldots, n$. That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

A squared matrix A $(n = m)$ is symmetric if $a_{ij} = a_{ji}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Let A be a squared matrix of size $n \times n$ (of **order** n) with $n \geq 2$. We denote by $\det(A)$ or $\mid A \mid$ the determinant of A.

Let A_{ij} be the matrix obtained by removing the ith row and jth column of A. To compute the determinant of A we can develop the determinant using the ith row of A

$$\det(A) = (-1)^{i+1} a_{i1} \mid A_{i1} \mid + (-1)^{i+2} a_{i2} \mid A_{i2} \mid + \cdots + (-1)^{i+n} a_{in} \mid A_{in} \mid$$

or the jth column of A

$$\det(A) = (-1)^{1+j} a_{1j} \mid A_{1j} \mid + (-1)^{2+j} a_{2j} \mid A_{2j} \mid + \cdots + (-1)^{n+j} a_{nj} \mid A_{nj} \mid .$$

Let A be an $m \times n$ matrix. The rank of A is the order of the biggest squared submatrix of A with non zero determinant. We will write $\operatorname{rank}(A)$.

2.SYSTEMS OF LINEAR EQUATIONS

Given a system of m linear equations in the variables $x_1, x_2, \ldots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{12}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $m \geq 1$, the coefficients a_{ij} and the independent terms b_i, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are real numbers.

The **matrix equation** of the system is $AX = b$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

where A is the the coefficient matrix, b the matrix of independent terms and X the matrix of unknowns.

The matrix of the system is

$$A \mid b = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

A solution of the system is a list $(s_1, s_2, \ldots, s_n)$ of real numbers that transforms the equations in identities when substituting the values $x_1, x_2, \ldots, x_n$ by $s_1, s_2, \ldots, s_n$ respectively.

The solution set of the system is the subset of $\mathbb{R}^n$ given by

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n \mid AS = b\} \text{ where } S = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}.$$

Rouche Theorem Let us consider a system of linear equations in n unknowns with matrix equation $AX = b$. The system

1. has a solution $\Leftrightarrow \operatorname{rank}(A \mid b) = \operatorname{rank}(A)$,
2. has a unique solution $\Leftrightarrow \operatorname{rank}(A \mid b) = \operatorname{rank}(A) = n$.

A system of linear equations with matrix equation $AX = b$ is homogeneous if b is a matrix of zeros. We write $AX = 0$.

A homogenous system of linear equations has always the zero solution and by the Rouche Theorem it has a nonzero solution (and therefore infinitely many solutions) if and only if $\operatorname{rank}(A \mid b) = \operatorname{rank}(A) < n$.

3. VECTOR SPACES AND SUBSPACES

A real vector space is a nonempty set V (of elements called **vectors**) where two operations are defined.

1. Addition of vectors $+ : V \times V \longrightarrow V$ is an internal operation: $\forall \mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$. Also $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ the next properties are verified:

   a) Commutative. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

   b) Distributive. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

   c) Zero element. There exists a zero vector 0_V in V such that $u+0_V = u$.

   d) Opposite element. $\forall u \in V$, $\exists -u \in V$ such that $u + (-u) = 0_V$.

2. Multiplication by scalars (real numbers) $\cdot : \mathbb{R} \times V \longrightarrow V$ is an external operation: $\forall \mathbf{v} \in V$ and $\forall a \in \mathbb{R}$ then $a\mathbf{v} \in V$. Also $\forall u, v \in V$, $\forall a, b \in \mathbb{R}$ it is verified:

   a) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

   b) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

   c) $(ab)\mathbf{u} = a(b\mathbf{u})$. d) $1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces

1. $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) | a_i \in \mathbb{R}, i = 1, \ldots n\}$, $n \in \mathbb{N}$ is a real vector space with the operations:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$\alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

with $(b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. The zero vector is $0_{\mathbb{R}^n} = (0, 0, \ldots, 0)$. The opposite vector of $(a_1, a_2, \ldots, a_n)$ is $(-a_1, -a_2, \ldots, -a_n)$.

2. Given a homogeneous system of linear equations with real coefficients

$$(*) \begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases}$$

with n unknowns. The set of solutions of the system $(*)$:

$$W = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n | (a_1, a_2, \ldots, a_n) \text{ is a solution of } (*)\} \subseteq \mathbb{R}^n$$

is a real vector space.

3. The set of matrices of size $m \times n$ (m rows and n columns) with real elements $\mathcal{M}_{m\times n}(\mathbb{R})$ is a real vector space with the sum of matrices and the product of matrices by scalars.

4. The set of the polynomials in x of degree less than or equal to n with real coefficients $\mathbb{R}_n[x]$ is a real vector space with the sum of polynomial and the product of polynomials by scalars.

   A polynomial $p(x)$ in $\mathbb{R}_n[x]$ is

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

   with $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

Definition of vector subspace

Let V be a real vector space.

A nonempty subset U of V is a vector subspace of V if $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \mathbf{u}, \mathbf{v} \in U$ it holds:

$$\alpha \mathbf{u} + \beta \mathbf{v} \in U.$$

Equivalently, a nonempty subset U of V is a vector subspace of V if it holds:

1. $\forall \mathbf{u}, \mathbf{v} \in U$, $\mathbf{u} + \mathbf{v} \in U$, thus $+$ is an inner operation in U,
2. $\forall \mathbf{u} \in U$, $\forall \alpha \in \mathbb{R}$, $\alpha \mathbf{u} \in U$, thus $\cdot$ is an external operation on U.

Therefore, U is a vector subspace if it is a vector space with the operations of V.

Examples of vector subspaces

1. Given a vector space V, the sets $\{0_V\}$ and V are vector subspaces of V.
2. $V = \mathbb{R}^3$, $U = \{(x, y, 0) | x, y \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^3$.
3. The solution set of a homogeneous system of linear equations in n unknowns with real coefficients is a vector subspace of $\mathbb{R}^n$.
4. Given a nonzero vector u in V, the set

$$U = \{au \mid a \in \mathbb{R}\}$$

is a vector subspace of V.

Let V be a real vector space.

A vector $u \in V$ is a linear combination of the vectors $u_1, \ldots, u_n \in V$ if there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$u = a_1 u_1 + \ldots + a_n u_n = \sum_{i=1}^{n} a_i u_i.$$

Let C be a nonempty subset of V. Then the set of all the linear combinations with vectors of C,

$$\langle C \rangle = \{ \sum_{i=1}^{n} a_i u_i \mid a_i \in \mathbb{R}, u_i \in C \}$$

is a vector subspace of V which is called the subspace generated by C.

Let $C = \{(1, 0, 0), (0, 1, 1)\} \subseteq \mathbb{R}^3$. Then

$$\langle C \rangle = \{(a, b, b) \mid a, b \in \mathbb{R}\}.$$

The intersection of two subspaces U_1 and U_2 of V

$$U_1 \cap U_2 = \{v \mid v \in U_1 \text{ and } v \in U_2\}.$$

is a vector subspace of V.

The subspace $\langle C \rangle$ generated by C verifies:

1. $C \subseteq \langle C \rangle$.
2. Every subspace W such that $C \subseteq W$ verifies $\langle C \rangle \subseteq W$.
3. The subspace $\langle C \rangle$ coincides with the intersection of all subspaces containing C.

A vector space V is finitely generated if there exists a finite set of vectors G such that $V = \langle G \rangle$, the set G is said to be a generating set of V.

Let $V = \mathbb{R}^3$.

1. $G_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a generating set of $\mathbb{R}^3$, since every vector $(x_1,x_2,x_3) \in \mathbb{R}^3$ can be written as

$$(x_1,x_2,x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1).$$

2. Let us check that $G_2 = \{(1,1,1),(1,1,0),(1,0,0)\}$ is a generating set of $\mathbb{R}^3$. Given $(x_1,x_2,x_3) \in \mathbb{R}^3$, we wonder if there exist $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$ such that

$$(x_1,x_2,x_3) = \lambda_1(1,1,1) + \lambda_2(1,1,0) + \lambda_3(1,0,0),$$

equivalently if the following system in the variables λ_1, λ_2 and λ_3 has a solution

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = x_1 \\ \lambda_1 + \lambda_2 = x_2 \\ \lambda_1 = x_3. \end{cases}$$

Since the coefficient matrix has rank 3, then Rouche Theorem implies that for every $(x_1,x_2,x_3) \in \mathbb{R}^3$ the system has a solution.

3. Let us check that $G_3 = \{(1,1,1),(2,1,1),(0,0,1),(3,0,0)\}$ is a generating set of $\mathbb{R}^3$. Given $(x_1, x_2, x_3) \in \mathbb{R}^3$, we wonder if there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$(x_1, x_2, x_3) = \lambda_1(1,1,1) + \lambda_2(2,1,1) + \lambda_3(0,0,1) + \lambda_4(3,0,0),$$

equivalently if the following system with unknown variables λ_1, λ_2, λ_3 and λ_4 has a solution

$$\begin{cases} \lambda_1 + 2\lambda_2 + 3\lambda_4 = x_1 \\ \lambda_1 + \lambda_2 = x_2 \\ \lambda_1 + \lambda_2 + \lambda_3 = x_3. \end{cases}$$

The coefficient matrix has rank 3 so Rouche's Theorem allows to say that for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ the system has infinitely many solutions.

4. $G_4 = \{(1, 1, -3), (0, 0, 1)\}$ is not a generating set of $\mathbb{R}^3$. The reason is that the system

$$\begin{cases} \lambda_1 = x_1 \\ \lambda_2 = x_2 \\ -3\lambda_1 + \lambda_2 = x_3 \end{cases}$$

does not have a solution for every $(x_1, x_2, x_3) \in \mathbb{R}^3$. For example, if $(x_1, x_2, x_3) = (0, 1, 0)$ the system has no solution.

We observe in the previous examples that the real vector space $\mathbb{R}^3$ is finitely generated but the generating set is not unique. The natural question is,

how many vectors do we need to generate $\mathbb{R}^3$?