4. BASES AND DIMENSION

Definition Let $u_1, \ldots, u_n$ be n vectors in V. The vectors $u_1, \ldots, u_n$ are **linearly independent** if the only linear combination of them equal to the zero vector has only zero scalars; that is, given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

$$\text{if } \lambda_1 u_1 + \ldots + \lambda_n u_n = 0_V \Rightarrow \lambda_1 = \ldots = \lambda_n = 0.$$

Otherwise, it is said that $u_1, \ldots, u_n$ are **linearly dependent**, that is

$$\exists \lambda_1, \ldots, \lambda_n \in \mathbb{R} \text{ not all zero such that } \lambda_1 u_1 + \ldots + \lambda_n u_n = 0_V.$$

1. Let us prove that $u_1 = (1,0,0)$, $u_2 = (0,-3,0)$, $u_3 = (0,0,5)$ are linearly independent in $\mathbb{R}^3$. If $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0_{\mathbb{R}^3}$ then $(\lambda_1, -3\lambda_2, 5\lambda_3) = (0,0,0)$ and thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
2. $\{(1,2,0),(2,0,0),(0,1,0),(0,0,6)\}$ is a set of linearly dependent vectors because $(1,2,0) - (1/2)(2,0,0) - 2(0,1,0) + 0(0,0,6) = (0,0,0)$.
3. If 0_V is an element of C, then C is a set of linearly dependent vectors.
4. $u, v \in V$ are linearly dependent $\Leftrightarrow u = \alpha v$, for some $\alpha \in \mathbb{R}$.

Theorem Let V be a real vector space. The number of elements of any generating set of V is greater than or equal to the number of elements of any set of linearly independent vectors of V.

Definition Let V be a finitely generated real vector space. A subset $B = \{v_1, \dots, v_n\}$ of V is a **basis** of V if it verifies,

1. B is a generating set of V,
2. B is a set of linearly independent vectors.

Example Let e_i be the vector of $\mathbb{R}^n$ with zeros in every entry except for the ith entry, which equals 1. $B = \{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$ called the **standard basis**.

Theorem (Existence of basis) Every set of generating vectors G of a real vector space V which is finitely generated and nonzero contains a basis B of V. Therefore, every real vector space V finitely generated and nonzero has a basis.

Example In $V = \mathbb{R}^2$ the set

$$G = \{(1,0),(1,1),(-1,0),(1,2)\}$$

is a generating set of V and contains the basis $B = \{(1,0),(1,1)\}$ of V, which is obtained by removing vectors of G linearly dependent of the remaining vectors of G. In this case $(-1,0) = (-1)(1,0)$ and $(1,2) = 2(1,1) - (1,0)$.

Proposition Let V be a finitely generated real vector space and let $B = \{v_1, \ldots, v_n\}$ be a basis of V. Every vector of V has a unique expression as a linear combination of the vectors of B.

Definition Let V be a finitely generated real vector space and let $B = \{v_1, \ldots, v_n\}$ be a basis of V. The coordinates of $v \in V$ are the scalars in the unique list $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$. We will write $(\lambda_1, \ldots, \lambda_n)_B$ to mean the coordinates of a vector in the basis B.

Example In $V = \mathbb{R}^3$ fix the basis $B = \{v_1 = (1,0,0), v_2 = (0,2,0), v_3 = (0,0,-1)\}$. The coordinates of $v = (2,1,1)$ in B are $(2,1/2,-1)_B$ sinse $v = 2v_1 + 1/2v_2 - 1v_3$.

Remarks

1. Observe that a vector space has infinitely many bases.
2. If V is a finitely generated vector space then every basis has a finite number of vectors.
3. The set $\mathbb{R}[x]$ of all the polynomials in x with real coefficients is a real vector space, which is not finitely generated. A basis of $\mathbb{R}[x]$ has infinitely many vectors.

Dimension Theorem All the bases of a finitely generated real vector space $V \neq \{0_V\}$ have the same number of vectors.

Definition The dimension of a finitely generated real vector space V, is the number of elements of a basis of V. We denote it by $\dim V$. By convention $\dim\{0_V\} = 0$.

Theorem Let V be a finitely generated real vector space. Every linearly independent set of vectors of V is included in a basis of V.

Example Let $V = \mathbb{R}^3$. The set of linearly independent vectors $I = \{(1,1,0),(0,2,0)\}$ is a subset of the basis

$$\{(1,1,0),(0,2,0),(0,0,1)\}$$

of $\mathbb{R}^3$, the vector $(0,0,1)$ was added to complete I to a basis of V.

Definition The rank of a set of vectors $C = \{u_1, \ldots, u_n\}$ in V is the dimension of the subspace they generate:

$$\operatorname{rank}(C) = \dim\langle C\rangle.$$

Given a finite dimensional vector space V, we fix a basis B. Let M be the matrix whose rows are the coordinates of the vectors of C in the basis B. Then,

$$\operatorname{rank}(C) = \operatorname{rank}(M).$$

Proposition The rank of a matrix M equals the highest number of row vectors of M (equivalently column vectors) which are linearly independent.

5. EQUATIONS OF SUBSPACES

Let V be a real vector space and let us fix a basis $B = \{v_1, \ldots, v_n\}$ of V. Let U be a vector subspace of V such that $0 < \mathrm{dim}U < \mathrm{dim}V$.

CARTESIAN EQUATIONS

Proposition There exists a homogeneous system of linear equations $AX = 0$, with $\dim V - \dim U$ equations, whose solution set equals the set of the coordinates of all the vectors of U in the basis B, that is

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 v_1 + \dots + \lambda_n v_n \in U\} =$$

$$= \{(s_1, s_2, \dots, s_n) \in \mathbb{R}^n \mid AS = 0\} \text{ where } S = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}.$$

Every homogeneous system verifying the previous statement is given the name of system of implicit or cartesian equations of U in the basis B.

Let us suppose that $\dim U = m$ (with $0 < m < n$) and let $\{u_1, \dots, u_m\}$ be a basis of U. To find the implicit equations of U means to look for conditions on the coordinates $(x_1, x_2, \dots, x_n)_B$ of a vector v in B so that v belongs to U.

Let us suppose that the coordinates of u_i in the basis B are $(u_{i1}, \ldots, u_{in})_B$ and built the matrix M of size $(m+1) \times n$:

$$M = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{pmatrix}.$$

The vector v belongs to U if it is a linear combination of the vectors in $\{u_1, \ldots, u_m\}$, then $\operatorname{rank}(M) = m$. This means that all the minors of order $m+1$ of M are zero. Each minor of order $m+1$ provides a homogeneous linear equation. Since $\dim U = m$ we can reduce the system to $l = n - m$ equations

$$\text{Cartesian equations of } U \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln}x_n = 0, \end{cases}$$

taking l minors of order $m+1$ containing a nonzero minor of order m previously fixed.

Observe that the cartesian equations of a subspace U in the basis B are not unique.

PARAMETRIC EQUATIONS

Definition The parametric equations of U in the basis B are a parametric solution of the system of cartesian equations of U in the basis B.

If $\dim U = m$ the parametric equations are

$$\text{Parametric equations of } U \left\{ \begin{array}{c} x_1 = u_{11}\alpha_1 + u_{21}\alpha_2 + \cdots + u_{m1}\alpha_m \\ x_2 = u_{12}\alpha_1 + u_{22}\alpha_2 + \cdots + u_{m2}\alpha_m \\ \vdots \\ x_n = u_{1n}\alpha_1 + u_{2n}\alpha_2 + \cdots + u_{mn}\alpha_m, \end{array} \right.$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the parameters and $u_{ij} \in \mathbb{R}$.

They can be also written as

$$
\begin{aligned}
&(x_1, x_2, \ldots, x_n) = \\
&= (u_{11}\alpha_1 + u_{21}\alpha_2 + \cdots + u_{m1}\alpha_m, \ \ldots, \ u_{1n}\alpha_1 + u_{2n}\alpha_2 + \cdots + u_{mn}\alpha_m).
\end{aligned}
$$

This means that a vector $v \in V$, with coordinates $(x_1, x_2, \ldots, x_n)_B$, belongs to U if

$$
\begin{aligned}
&(x_1, x_2, \ldots, x_n)_B = \\
&= (u_{11}\alpha_1 + u_{21}\alpha_2 + \cdots + u_{m1}\alpha_m, \ \ldots, \ u_{1n}\alpha_1 + u_{2n}\alpha_2 + \cdots + u_{mn}\alpha_m)_B = \\
&= \alpha_1(u_{11}, \ldots, u_{1n})_B + \alpha_2(u_{21}, \ldots, u_{2n})_B \ldots + \alpha_m(u_{m1}, \ldots, u_{mn})_B.
\end{aligned}
$$

If we call u_i the coordinate vector $(u_{i1}, \ldots, u_{in})_B$, $i = 1, \ldots, m$ then

$$\{u_1, u_2, \ldots, u_m\}$$

is a basis of U.

6. OPERATIONS WITH SUBSPACES

INTERSECTION OF SUBSPACES

Let V be a real vector space with basis B.
The intersection of vector subspaces is a vector subspace. We explain next how to obtain the equations of the intersection of two subspaces.

Let U_1 and U_2 be vector subspaces of V with systems of cartesian equations

$$\text{Equations of } U_1 \left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{l_1 1}x_1 + a_{l_1 2}x_2 + \cdots + a_{l_1 n}x_n = 0, \end{array} \right.$$

$$\text{Equations of } U_2 \left\{ \begin{array}{c} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n = 0 \\ \vdots \\ b_{l_2 1}x_1 + b_{l_2 2}x_2 + \cdots + b_{l_2 n}x_n = 0, \end{array} \right.$$

U_1 has l_1 equations and U_2 has l_2 equations.

The intersection subspace $U_1 \cap U_2$ contains those vectors of V whose coordinates $(x_1, x_2, \ldots, x_n)_B$ verify the system:

$$(*) = \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{l_1 1}x_1 + a_{l_1 2}x_2 + \cdots + a_{l_1 n}x_n = 0 \\ b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n = 0 \\ \vdots \\ b_{l_2 1}x_1 + b_{l_2 2}x_2 + \cdots + b_{l_2 n}x_n = 0, \end{cases}$$

Remarks

1. The system of cartesian equations of $U_1 \cap U_2$ is obtained removing from $(*)$ equations that depend linearly on the remaining equations. This can be achieved obtaining the echelon form of $(*)$. Thus, the number of cartesian equations of $U_1 \cap U_2$ is $\leq l_1 + l_2$.

2. A parametric solution of $(*)$ provides the parametric equations of $U_1 \cap U_2$ and from them a basis of $U_1 \cap U_2$ is obtained.

SUM OF SUBSPACES

Given subspaces U_1 and U_2 of V, in general $U_1 \cup U_2$ is not a subspace of V. Let us see a counterexample.

Example Given the subspaces of $V = \mathbb{R}^2$

$$U_1 = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$$
$$U_2 = \{(0, y) \in \mathbb{R}^2 | y \in \mathbb{R}\}.$$

The set $U_1 \cup U_2$ is not a subspace of $\mathbb{R}^2$ since $u_1 = (1, 0) \in U_1$, $u_2 = (0, 1) \in U_2$ but $u_1 + u_2 = (1, 1) \notin U_1 \cup U_2$.

Definition Let U_1 and U_2 be subspaces of V. We call sum of U_1 and U_2 to the set

$$U_1 + U_2 = \{u_1 + u_2 | u_1 \in U_1, u_2 \in U_2\}.$$

Proposition The set $U_1 + U_2$ is a vector subspace of V, it is the smallest subspace that contains $U_1 \cup U_2$, that is $U_1 + U_2 = \langle U_1 \cup U_2 \rangle$.

To compute $U_1 + U_2$ it is important to take into account that if B_{U_1} is a basis of U_1 and B_{U_2} is a basis of U_2 then

$$U_1 + U_2 = \langle B_{U_1} \cup B_{U_2} \rangle,$$

that is, $B_{U_1} \cup B_{U_2}$ is a generating set of $U_1 + U_2$. We can obtain a basis $B_{U_1+U_2}$ of $U_1 + U_2$ from $B_{U_1} \cup B_{U_2}$ by removing vectors, to obtain a linearly independent set, which is also a generating set.

DIRECT SUM OF SUBSPACES

Definition Let U_1 and U_2 be subspaces of V. The subspace $U_1 + U_2$ is a direct sum if

$$U_1 \cap U_2 = \{0_V\}.$$

We write $U_1 \oplus U_2$.

Example Let $V = \mathbb{R}^3$ and fix the standard basis. Let us consider vector subspaces U and W given by their cartesian equations,

$$U \equiv \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_3 = 0, \end{cases} \quad W \equiv 2x_2 + x_3 = 0.$$

The coordinates (x_1, x_2, x_3) of a vector of $U \cap W$ are a solution of the system

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \\ 2x_2 + x_3 = 0, \end{cases}$$

whose coefficient matrix has rank 3. By Rouche's Theorem the system has only the zero solution (it is a homogeneous system). This shows that $U \cap W = \{0_{\mathbb{R}^3}\}$ and then $U \oplus W$, that is the sum is a direct sum.

Proposition Let B_1 be a basis of U_1 and B_2 a basis of U_2 then:

$$U_1 \cap U_2 = \{0_V\} \Leftrightarrow B_1 \cup B_2 \text{ is a linearly independent set,}$$

that is

$$U_1 \oplus U_2 \Leftrightarrow B_1 \cup B_2 \text{ is a basis of } U_1 + U_2.$$

Definition Two subspaces U_1 and U_2 of a real vector space V are complementary if $U_1 \oplus U_2 = V$.

$$U_1 \oplus U_2 = V \Leftrightarrow \begin{cases} U_1 + U_2 = V \\ U_1 \cap U_2 = \{0_V\} \end{cases}$$

Remarks

1. Every subspace has a complementary subspace.
2. Let U_1 and U_2 be complementary subspaces of V. If B_1 is a basis of U_1 and B_2 is a basis of U_2 then $B_1 \cup B_2$ is a basis of V.

Theorem Dimension Formula or Grassman Formula Let U_1 and U_2 be subspaces of V. Then

$$\dim U_1 + \dim U_2 = \dim(U_1 + U_2) + \dim(U_1 \cap U_2).$$