

## 8. DIAGONALIZACION

## INTRODUCTION

The squared matrix

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

and the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

verify

$$D = P^{-1}AP \text{ where } P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let  $V$  be a real vector space.

If  $f : V \rightarrow V$  is an endomorphism, whose matrix  $D$  in some basis of  $V$  is diagonal, then numerous problems related with  $f$  are significantly simplified.

Some examples are: classify  $f$  (injective, surjective, bijective); obtain its invariants; compute  $f^n$ ,  $n \in \mathbb{N}$ .

Given a squared matrix  $A$  (associated to an endomorphism) we explain next how to obtain a diagonal matrix  $D$  related with  $A$  by

$$D = P^{-1}AP \text{ where } P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

# EIGENVALUES AND EIGENVECTORS OF AN ENDOMORPHISM

Let  $V$  be a nonzero real vector space and  $f : V \rightarrow V$  an endomorphism.

**Definition** A scalar  $\lambda \in \mathbb{R}$  is an **eigenvalue** of  $f$  if there exists a nonzero vector  $v \in V$  such that

$$f(v) = \lambda v.$$

**Definition** If  $\lambda$  is an eigenvalue of  $f$ , a vector  $v \in V$  verifying  $f(v) = \lambda v$  is called **eigenvector** of  $f$  associated to  $\lambda$ .

**Example** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the endomorphism defined by

$$f(x, y, z) = (x + 3y + 3z, -3x - 5y - 3z, 3x + 3y + z).$$

Then,  $\lambda = -2$  is an eigenvalue of  $f$  because there exists a nonzero vector  $v = (-1, 1, 0)$  such that  $f(v) = -2v$ . Thus  $v$  is an eigenvector of  $f$  associated to  $\lambda = -2$ .

**Proposition** The set of all the eigenvectors of  $f$  associated to the same eigenvalue  $\lambda$  of  $f$

$$V_\lambda = \{v \in V \mid f(v) = \lambda v\}$$

is a vector subspace of  $V$  called **eigenspace** of  $f$  associated to  $\lambda$ .

**Remarks** Let  $id : V \rightarrow V$  be the identity map in  $V$ .

1.  $V_\lambda = \text{Ker}(f - \lambda id)$ .
2.  $\lambda \in \mathbb{R}$  is an eigenvalue of  $f \Leftrightarrow$  the endomorphism  $f - \lambda id$  is not one-to-one.
3. In particular,  $\lambda = 0$  is an eigenvalue of  $f \Leftrightarrow f$  is not one-to-one  $\Leftrightarrow \text{Ker}(f) = V_0 \neq \{0_V\}$ .
4. Some endomorphisms do not have eigenvalues, so they do not have eigenvectors either.

**Proposition** Let  $\lambda_1, \dots, \lambda_p$  be  $p$  different eigenvalues of  $f$ .

1. Let  $v_i$  be a nonzero eigenvector of  $f$  associated to  $\lambda_i$ ,  $i = 1, \dots, p$  then  $v_1, \dots, v_p$  are linearly independent.
2.  $V_{\lambda_1} + \dots + V_{\lambda_p}$  is a direct sum.

**Remark** If  $\dim V = n$  then the endomorphism  $f$  will have at most  $n$  different eigenvalues, otherwise it would have more than  $n$  linearly independent eigenvectors.

# EIGENVALUES AND EIGENVECTORS OF A SQUARED MATRIX

Let  $A$  be a squared  $n \times n$  matrix whose entries are real numbers, that is  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ .

**Definition** A scalar  $\lambda \in \mathbb{R}$  is an **eigenvalue** of  $A$  if there exists a nonzero column vector  $X \in \mathcal{M}_{n \times 1}(\mathbb{R})$  such that

$$AX = \lambda X.$$

**Definition** If  $\lambda$  is an eigenvalue of  $A$ , a column vector  $X \in \mathcal{M}_{n \times 1}(\mathbb{R})$  verifying  $AX = \lambda X$  is called an **eigenvector** of  $A$  associated to  $\lambda$ .

Let  $I_n$  be the identity matrix of size  $n \times n$ .

**Remarks** Let us assume that  $\dim V = n$  and let  $B$  be a basis of  $V$ . Let  $f : V \rightarrow V$  be the endomorphism whose matrix in the basis  $B$  is  $A$ . The following statements are verified:

1.  $\lambda$  is an eigenvalue of  $f \Leftrightarrow \lambda$  is an eigenvalue of  $A \Leftrightarrow \det(A - \lambda I_n) = 0$ .
2. Let  $v \in V$  and let  $X$  be the column vector of the coordinates of  $v$  in the basis  $B$ , that is  $X \in \mathcal{M}_{n \times 1}(K)$ . Then,  $v$  is an eigenvector of  $f \Leftrightarrow X$  is an eigenvector of  $A \Leftrightarrow (A - \lambda I_n)X = 0$ .
3.  $\dim V_\lambda = n - \text{rank}(A - \lambda I_n)$ .

**Example** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the endomorphism defined by

$$f(x, y, z) = (3x, x + 2y, 4x + 2z).$$

Let  $B$  be the standard basis of  $\mathbb{R}^3$ . The matrix of  $f$  in the basis  $B$  is

$$A = M_f(B) = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{pmatrix}.$$

Since  $\text{rank}(A - 3I_3) = 2$  the system  $(A - 3I_3)X = 0$ , that is

$$\begin{pmatrix} 3 - 3 & 0 & 0 \\ 1 & 2 - 3 & 0 \\ 4 & 0 & 2 - 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nonzero solution. Therefore,  $\lambda = 3$  is an eigenvalue of  $f$  and  $A$ . The eigenspace  $V_3$  has  $\dim V_3 = 1$  and cartesian equations

$$\text{Cartesian equations of } V_3 \begin{cases} x + y = 0 \\ 4x - z = 0 \end{cases}.$$

Thus  $V_3 = \{(a, a, 4a) \mid a \in \mathbb{R}\}$ .

On the other hand,  $\text{rank}(A - 2I_3) = 1$ , so  $\lambda = 2$  is an eigenvalue of  $f$  and  $A$ ,  $\dim V_2 = 2$ . Solving  $(A - 2I_3)X = 0$ , that is

$$\begin{pmatrix} 3-2 & 0 & 0 \\ 1 & 2-2 & 0 \\ 4 & 0 & 2-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we have  $V_2 = \{(0, a, b) \mid a, b \in \mathbb{R}\}$  and the cartesian equation of  $V_2$  is  $x = 0$ .

## ROOTS OF POLYNOMIALS AND THEIR MULTIPLICITY

Let  $\mathbb{R}[x]$  be the set of all polynomials in  $x$  with real coefficients.

**Definition** Let  $p(x)$  be a polynomial in  $\mathbb{R}[x]$ . A scalar  $\lambda \in \mathbb{R}$  is a **root** of  $p(x)$  if  $p(\lambda) = 0$ .

Equivalently,  $\lambda$  is a root of  $p(x)$  if and only if  $(x - \lambda)$  divides  $p(x)$ , that is, there exists a polynomial  $q(x) \in \mathbb{R}[x]$  such that

$$p(x) = (x - \lambda)q(x).$$

**Definition** Let  $\lambda$  be a root of the polynomial  $p(x)$ . We call **multiplicity** of  $\lambda$  to the highest natural number  $m$  such that  $(x - \lambda)^m$  divides  $p(x)$ , so

$$p(x) = (x - \lambda)^m q(x), \quad q(x) \in \mathbb{R}[x].$$

Every polynomial of degree greater or equal than one has all its roots in  $\mathbb{C}$ . On the other hand, there exist polynomials in  $\mathbb{R}[x]$  with no roots in  $\mathbb{R}$ . As an example,  $x^2 + 1$  has real coefficients but only complex roots.

**Proposition** Let  $p(x) \in \mathbb{R}[x]$  be a polynomial of degree  $n$  whose real roots are  $\lambda_1, \dots, \lambda_p$  with multiplicities  $m_1, \dots, m_p$  respectively. Then:

1. There exists  $q(x) \in \mathbb{R}[x]$  such that  $p(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_p)^{m_p} q(x)$  so  $m_1 + \cdots + m_p \leq n$ .
2. If  $p(x) \in \mathbb{C}[x]$  all its roots are in  $\mathbb{C}$  so

$$p(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_p)^{m_p} \text{ with } m_1 + \cdots + m_p = n.$$

# CHARACTERISTIC POLYNOMIAL

**Definition** Let  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ . The **characteristic polynomial** of  $A$  is  $\det(A - \lambda I_n)$  and its **characteristic equation** is

$$\det(A - \lambda I_n) = 0.$$

**Proposition** Let  $A$  and  $A'$  be matrices in  $\mathcal{M}_{n \times n}(\mathbb{R})$ . If  $A$  and  $A'$  are matrices of the same endomorphism in different bases then they have the same characteristic polynomial.

Let  $V$  be a nonzero real vector space and let  $f : V \rightarrow V$  be an endomorphism.

**Remark**

1. All the matrices associated to  $f$  in different bases of  $V$  have the same characteristic polynomial.
2. The converse of the previous proposition is not true.

**Definition** The characteristic polynomial of  $f$  is the characteristic polynomial of any of the matrices associated to  $f$  in the different bases of  $V$ . Analogously with the characteristic equation of  $f$ .

**Example** Let us compute the characteristic polynomial of the endomorphism  $f$  of  $\mathbb{R}^5$  defined by

$$f(x_1, x_2, x_3, x_4, x_5) = (-x_2, x_1 + x_3, 2x_3 - x_4, 2x_4 + 6x_5, 3x_5).$$

Let  $B$  be the standard basis of  $\mathbb{R}^5$  and  $A = M_f(B)$ , then:

$$\begin{aligned}
 \det(A - \lambda I_5) &= \begin{vmatrix} 0 - \lambda & -1 & 0 & 0 & 0 \\ 1 & 0 - \lambda & 1 & 0 & 0 \\ 0 & 0 & 2 - \lambda & -1 & 0 \\ 0 & 0 & 0 & 2 - \lambda & 6 \\ 0 & 0 & 0 & 0 & 3 - \lambda \end{vmatrix} = \\
 &= (3 - \lambda)(2 - \lambda)^2 \begin{vmatrix} 0 - \lambda & -1 \\ 1 & 0 - \lambda \end{vmatrix} \\
 &= (3 - \lambda)(2 - \lambda)^2(\lambda^2 + 1).
 \end{aligned}$$

# MULTIPLICITY OF AN EIGENVALUE

Let  $V$  be a nonzero real vector space and  $f : V \rightarrow V$  an endomorphism. Let  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ .

**Definition** Let  $\lambda$  be an eigenvalue of  $f$  (or  $A$ ). We call **multiplicity** of  $\lambda$  to its multiplicity as a root of the characteristic equation of  $f$  (or  $A$ ).

**Theorem** Let  $\lambda$  be an eigenvalue of  $f$  (or  $A$ ) with multiplicity  $m$ . Then

$$1 \leq \dim V_\lambda \leq m.$$

## Remarks

1. Let  $\lambda$  be an eigenvalue of  $f$  (or  $A$ ) with multiplicity  $m$ . If  $m = 1$  then  $\dim V_\lambda = 1$
2. If  $A, A' \in \mathcal{M}_{n \times n}(\mathbb{R})$  are matrices of the same endomorphism in different bases then they have the same eigenvalues, with the same multiplicities and the same dimensions of their eigenspaces.

**Proposition** Let us suppose that  $\dim V = n$ . Let  $\lambda_1, \dots, \lambda_p$  be the distinct eigenvalues of  $f$  (or  $A$ ),  $m_1, \dots, m_p$  their multiplicities and  $d_1, \dots, d_p$  the dimensions of the corresponding eigenspaces. Then the maximum number of linearly independent eigenvectors of  $f$  (or  $A$ ) is  $d_1 + \dots + d_p$ . Furthermore,

$$p \leq d_1 + \dots + d_p \leq m_1 + \dots + m_p \leq n.$$

**Remark** The characteristic polynomial may not have only real roots and then  $m_1 + \dots + m_p < n$ . In fact, it could have no real roots and therefore no eigenvalues in this case.

**Example** We obtain next the eigenvalues of the matrix  $A$  together with their multiplicities and the dimensions of the eigenspaces,

$$A = \begin{pmatrix} 1 & 2 & 10 \\ 2 & 1 & 10 \\ -1 & -1 & -6 \end{pmatrix}.$$

The characteristic polynomial is  $\det(A - \lambda I_3) = \lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 2)(\lambda + 1)^2$ . Then we have eigenvalues  $\lambda_1 = -2$  with multiplicity  $m_1 = 1$  and  $\lambda_2 = -1$  with multiplicity  $m_2 = 2$ . The dimensions are

$$d_1 = \dim V_{\lambda_1} = 3 - \text{rank}(A - \lambda_1 I_3) = 1,$$

$$d_2 = \dim V_{\lambda_2} = 3 - \text{rank}(A - \lambda_2 I_3) = 2.$$

# DIAGONALIZATION OF ENDOMORPHISMS AND SQUARED MATRICES

Let  $V$  be a nonzero real vector space and let  $f : V \rightarrow V$  be an endomorphism. Let  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ .

**Definition** The endomorphism  $f$  is **diagonalizable** if there exists a basis  $B'$  of  $V$  so that the matrix  $M_f(B')$  is diagonal. Then, to **diagonalize**  $f$  is to find  $B'$ .

**Definition** The matrix  $A$  is **diagonalizable** if there exists a diagonal matrix  $D$  and an invertible matrix  $P \in \mathcal{M}_{n \times n}(\mathbb{R})$  such that  $D = P^{-1}AP$ . Then, to **diagonalize**  $A$  means to find  $D$  and  $P$ .

## Remarks

1. Let us suppose that  $A$  is the matrix of  $f$  in a basis  $B$ . Then:

$f$  is diagonalizable  $\Leftrightarrow A$  is diagonalizable.

2. If  $D = P^{-1}AP$  is a diagonalization of  $A$  then

a)  $D = M_f(B')$  is the matrix of  $f$  in a basis  $B'$  of  $V$  consisting of eigenvectors.

b)  $P = M(B', B)$  is the matrix of the change of coordinates from  $B'$  to  $B$ .

**Proposition** An endomorphism  $f$  is diagonalizable if and only if there exists a basis of  $V$  consisting of eigenvectors of  $f$ .

**Theorem** Let us suppose that  $\dim V = n$ . Let  $\lambda_1, \dots, \lambda_p$  be the distinct eigenvalues of  $f$  (or  $A$ ),  $m_1, \dots, m_p$  their multiplicities and  $d_1, \dots, d_p$  the dimensions of the eigenspaces. The following are necessary and sufficient conditions for the existence of a basis of  $V$  consisting of eigenvectors.

1. The characteristic polynomial of  $f$  has only real roots

$$\lambda_1, \dots, \lambda_p$$

, that is

$$m_1 + \dots + m_p = n.$$

2. The multiplicity of each eigenvalue equals the dimension of its eigenspace, this is

$$m_i = d_i, \quad i = 1, \dots, p.$$

**Corollary** Let us suppose that  $f$  is diagonalizable. Let  $\lambda_1, \dots, \lambda_p$  be the eigenvalues of  $f$  with multiplicities  $m_1, \dots, m_p$  respectively. Let  $B_i$  be a basis of the eigenspace  $V_{\lambda_i}$  having  $m_i = d_i$  elements,  $i = 1, \dots, p$ . Then:

1.  $B' = B_1 \cup \dots \cup B_p$  is a basis of  $V$  consisting of eigenvectors of  $f$ .
2. The matrix of  $f$  in the basis  $B'$  is diagonal and its main diagonal contains the elements

$$\lambda_1, \overset{m_1}{\dots}, \lambda_1, \lambda_2, \overset{m_2}{\dots}, \lambda_2, \dots, \lambda_p, \overset{m_p}{\dots}, \lambda_p$$

**Examples** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the endomorphism defined by

$$f(x, y, z) = (x + 2y + 10z, 2x + y + 10z, -x - y - 6z)$$

whose matrix in the standard basis  $B$  of  $\mathbb{R}^3$  is a matrix  $A$  whose eigenvalues are  $\lambda_1 = -2$  and  $\lambda_2 = -1$ . A basis of  $V_{\lambda_1}$  is

$$B_{\lambda_1} = \{(-2, -2, 1)\}$$

and of  $V_{\lambda_2}$  is

$$B_{\lambda_2} = \{(-5, 0, 1), (-1, 1, 0)\}.$$

Then a basis of  $\mathbb{R}^3$  consisting of eigenvectors of  $f$  is

$$B' = B_{\lambda_1} \cup B_{\lambda_2} = \{(-2, -2, 1), (-5, 0, 1), (-1, 1, 0)\}.$$

Finally,

$$D = P^{-1}AP = M_f(B') = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where

$$P = \begin{pmatrix} -2 & -5 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$