CHAPTER II: AFFINE AND EUCLIDEAN GEOMETRY

1. AFFINE SPACE

1.1 Definition of affine space

A real affine space is a triple $(\mathbb{A}, V, \phi)$ where $\mathbb{A}$ is a set of points, V is a real vector space and $\phi\colon \mathbb{A} \times \mathbb{A} \longrightarrow V$ is a map verifying:

1. $\forall P \in \mathbb{A}$ and $\forall u \in V$ there exists a unique $Q \in \mathbb{A}$ such that
$$\phi(P, Q) = u.$$
2. $\phi(P, Q) + \phi(Q, R) = \phi(P, R)$ for every $P, Q, R \in \mathbb{A}$.

Notation. We will write $\phi(P, Q) = \overline{PQ}$. The elements contained in the set $\mathbb{A}$ are called points of $\mathbb{A}$ and we will say that V is the vector space associated to the affine space $(\mathbb{A}, V, \phi)$. We define the dimension of the affine space $(\mathbb{A}, V, \phi)$ as

$$\dim \mathbb{A} = \dim V.$$

Examples

1. Every vector space V is an affine space with associated vector space V. Indeed, in the triple $(\mathbb{A}, V, \phi)$, $\mathbb{A} = V$ and the map ϕ is given by

$$\phi\colon \mathbb{A} \times \mathbb{A} \longrightarrow V, \quad \phi(P, Q) = P - Q, \forall P, Q \in \mathbb{A}.$$

2. According to the previous example, $(\mathbb{R}^2, \mathbb{R}^2, \phi)$ is an affine space of dimension 2, $(\mathbb{R}^3, \mathbb{R}^3, \phi)$ is an affine space of dimension 3. In general $(\mathbb{R}^n, \mathbb{R}^n, \phi)$ is an affine space of dimension n.

Properties of affine spaces

Let $(\mathbb{A}, V, \phi)$ be a real affine space. The following statemens hold:

1. $\phi(P, Q) = 0$ if and only if $P = Q$.
2. $\phi(P, Q) = -\phi(Q, P)$, $\forall P, Q \in \mathbb{A}$.
3. $\phi(P, Q) = \phi(R, S)$ if and only if $\phi(P, R) = \phi(Q, S)$.

1.2 Affine coordinate system

Let $\mathbb{A}$ be an affine space of dimension n with associated vector space V.

Definition of affine coordinate system

A set of $n+1$ points $\{O, P_1, \ldots, P_n\}$ of an affine space $(\mathbb{A}, \mathbb{V}, \phi)$ is an affine coordinate system of $\mathbb{A}$ if the vector set $\{\overline{OP_1}, \ldots, \overline{OP_n}\}$ in a basis of the vector space V.

A point $O \in \mathbb{A}$ such that $B = \{\overline{OP_1}, \ldots, \overline{OP_n}\}$ is a basis of V, is called origin of the coordinate system $\{O, P_1, \ldots, P_n\}$.

Proposition Given a point $P_0 \in \mathbb{A}$ there exists an affine coordinate system of $\mathbb{A}$ in which P_0 is the origin.

A point $O \in \mathbb{A}$ and a basis B of V determine an affine coordinate system $\mathcal{R}$ of $\mathbb{A}$, we write $\mathcal{R} = \{O; B\}$.

Definition

We call coordinates of a point $P \in \mathbb{A}$ with respect to a cartesian coordinate system $\mathcal{R} = \{O; B\}$ of the affine space $\mathbb{A}$, to the coordinates of the vector $\overline{OP}$ with respect to the basis B of the vector space V; this is, the n-tuple $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{R}^n$ such that

$$\overline{OP} = \alpha_1 u_1 + \cdots + \alpha_n u_n,$$

where $u_1, \ldots, u_n$ are the vectors of the basis B. We write $P(\alpha_1, \ldots, \alpha_n)_{\mathcal{R}}$.

Example

Let $\mathcal{R} = \{O(0,0,0); B_c\}$ be an affine coordinate system of the affine space $(\mathbb{R}^3, \mathbb{R}^3, \phi)$ of dimension 3, where B_c is the standard basis of $\mathbb{R}^3$.
Let us consider the coordinate system $\mathcal{R}' = \{O'; B'\}$ with $O'(1,2,-1)$ and $B' = \{u_1, u_2, u_3\}$. The vectors $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$ form a basis of V as we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

so the vector set $\{u_1, u_2, u_3\}$ is linearly independent (we know that a linearly independent set with 3 vectors in a vector space V of dimension 3 is a basis).

Let P be a point with coordinates $(5,5,0)$ with respect to $\mathcal{R}$, this is,

$$P(5,5,0)_{\mathcal{R}} \Longleftrightarrow \overline{OP} = 5u_1 + 5u_2 + 0u_3.$$

We are going to calculate the coordinates of P with respect to $\mathcal{R}'$:

$$\begin{aligned}
\overline{O'P} &= (5-1, 5-2, 0+1) = (4,3,1), \\
\overline{O'P} &= x_1u_1 + x_2u_2 + x_3u_3 = x_1(1,0,0) + x_2(1,1,0) + x_3(1,1,1) \\
&= (x_1 + x_2 + x_3,\ x_2 + x_3,\ x_3),
\end{aligned}$$

thus

$$\begin{cases} 4 = x_1 + x_2 + x_3 \\ 3 = x_2 + x_3 \\ 1 = x_3 \end{cases} \Longrightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$

and so, $(1,2,1)$ are the coordinates of P with respect to $\mathcal{R}'$: $P(1,2,1)_{\mathcal{R}'}$.

Change of affine coordinates

We will limit our study to the case of an affine space of dimension 2.

Let $\mathbb{A}$ be an affine space of dimension 2 with associated vector space V. Let $B = \{u_1, u_2\}$ and $B' = \{u'_1, u'_2\}$ be two bases of V and $\mathcal{R} = \{O; B\}$, $\mathcal{R}' = \{O'; B'\}$ two affine coordinate systems of $\mathbb{A}$.

Let us consider $P \in \mathbb{A}$, and let (x_1, x_2) be the coordinates of P with respect to $\mathcal{R}$ and (x'_1, x'_2) the coordinates of P with respect to $\mathcal{R}'$; this is,

$$\begin{aligned} OP &= x_1u_1 + x_2u_2, \\ \text{and } O'P &= x'_1u'_1 + x'_2u'_2. \end{aligned}$$

What is the relationship between (x_1, x_2) and (x'_1, x'_2)?

We know that

$$\overline{OP} = \overline{OO'} + \overline{O'P}.$$

Let (a, b) be the coordinates of O' with respect to $\mathcal{R}$; this is,

$$\overline{OO'} = au_1 + bu_2,$$

and let

(a_{11}, a_{21}) be the coordinates of u'_1 with respect to the basis B,
(a_{12}, a_{22}) be the coordinates of u'_2 with respect to the basis B;

this is,

$$\begin{aligned} u'_1 &= a_{11}u_1 + a_{21}u_2, \\ u'_2 &= a_{12}u_1 + a_{22}u_2. \end{aligned}$$

If we substitute all this in $\overline{OP} = \overline{OO'} + \overline{O'P}$ we obtain:

$$\begin{aligned} \overline{OP} &= \overline{OO'} + \overline{O'P} \\ &= au_1 + bu_2 + x'_1u'_1 + x'_2u'_2 \\ &= au_1 + bu_2 + x'_1\left(a_{11}u_1 + a_{21}u_2\right) + x'_2\left(a_{12}u_1 + a_{22}u_2\right) \\ &= \left(a + x'_1a_{11} + x'_2a_{12}\right)u_1 + \left(b + x'_1a_{21} + x'_2a_{22}\right)u_2, \end{aligned}$$

and since $\overline{OP} = x_1u_1 + x_2u_2$, if we equate the coefficients we have:

$$\begin{cases} x_1 = a + x_1'a_{11} + x_2'a_{12} \\ x_2 = b + x_1'a_{21} + x_2'a_{22} \end{cases}$$

We can also write this equation system as a matrix equation:

$$\begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & a_{11} & a_{12} \\ b & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ x_1' \\ x_2' \end{pmatrix}.$$

Let us consider the general case. Let $\mathbb{A}$ be an n-dimensional affine space, and let $\mathcal{R} = \{O; B = \{u_1, \ldots, u_n\}\}$ and $\mathcal{R}' = \{O'; B' = \{u_1', \ldots, u_2'\}\}$ be two affine coordinate systems of $\mathbb{A}$.

Let $(x_1, \ldots, x_n)$ be the coordinates of P with respect to $\mathcal{R}$ and $(x'_1, \ldots, x'_n)$ the coordinates of P with respect to $\mathcal{R}'$ then we have:

$$\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ x'_1 \\ \vdots \\ x'_n \end{pmatrix}$$

where

$(a_1, \ldots, a_n)$ are the coordinates of O' with respect to $\mathcal{R}$,

$(a_{11}, \ldots, a_{n1})$ are the coordinates of u'_1 with respect to the basis B,

$\vdots$

$(a_{1n}, \ldots, a_{nn})$ are the coordinates of u'_n with respect to the basis B.

We can also write it as follows:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + A \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}$$

where A is the matrix of change of basis from B' to B:

$$A = M(B', B) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

The matrix

$$M(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is the change of coordinates matrix from $\mathcal{R}'$ to $\mathcal{R}$.

Example

In the affine space $(\mathbb{A}_2, V_2, \phi)$ we consider the coordinate systems $\mathcal{R} = \{O; B = \{u_1, u_2\}\}$, $\mathcal{R}' = \{O'; B' = \{u'_1, u'_2\}\}$ with

$$\overline{OO'} = 3u_1 + 3u_2, \quad u'_1 = 2u_1 - u_2, \quad u'_2 = -u_1 + 2u_2.$$

1. Determine the change of coordinates matrix from $\mathcal{R}'$ to $\mathcal{R}$.

   We have

$$\begin{aligned} \overline{OP} &= \overline{OO'} + \overline{O'P} = 3u_1 + 3u_2 + y_1(2u_1 - u_2) + y_2(-u_1 + 2u_2) \\ &= (3 + 2y_1 - y_2)\, u_1 + (3 - y_1 + 2y_2)\, u_2, \end{aligned}$$

   so

$$\begin{cases} x_1 = 3 + 2y_1 - y_2 \\ x_2 = 3 - y_1 + 2y_2 \end{cases}$$

   this is,

$$M(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & -1 \\ 3 & -1 & 2 \end{pmatrix}.$$

2. Determine the change of coordinates matrix from $\mathcal{R}$ to $\mathcal{R}'$.

$$M(\mathcal{R},\mathcal{R}') = M(\mathcal{R}',\mathcal{R})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & -1 \\ 3 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & \frac{2}{3} & \frac{1}{3} \\ -3 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

3. The coordinates of a point P with respect to the coordinate system $\mathcal{R}$ are $(3,5)$. Determine the coordinates of P in $\mathcal{R}'$.

$$M(\mathcal{R},\mathcal{R}') \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & \frac{2}{3} & \frac{1}{3} \\ -3 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}, \text{ thus } P\left(\frac{2}{3},\frac{4}{3}\right)_{\mathcal{R}}.$$

4. The coordinates of a point Q with respect to the coordinate system $\mathcal{R}'$ are $(2,3)$. Determine the coordinates of Q in $\mathcal{R}$.

$$M_f(\mathcal{R}',\mathcal{R}) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & -1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \text{ thus } Q(4,7)_{\mathcal{R}}.$$

1.3 Affine subspace

Definition of affine subspace

Let $(\mathbb{A}, V, \phi)$ be a real affine space. A subset $L \subset \mathbb{A}$ is an affine subspace of $\mathbb{A}$ if given a point $P \in L$ the set

$$W(L) = \{\overline{PQ} \mid Q \in L\}$$

is a vector subspace of V. The vector subspace $W(L)$ is called vector subspace associated to L and it is denoted by $\overline{L}$.

Proposition The previous definition does not depend on the point P.

Proposition Let $(\mathbb{A}, V, \phi)$ be a real affine space and L an affine subspace of $\mathbb{A}$. The triple $(L, \overline{L}, \phi)$ is an affine space.

Proposition Let $(\mathbb{A}, V, \phi)$ be a real affine space and L an affine subspace of $\mathbb{A}$. For every point $P \in \mathbb{A}$ and each vector subspace $W \subset V$ the set

$$L(P, W) = \{X \in \mathbb{A} \mid \overline{PX} \in W\}$$

is an affine subspace of $\mathbb{A}$ that we denote by $P + W$.

Definition of dimension of an affine subspace

Let $(\mathbb{A}, V, \phi)$ be a real affine space and L an affine subspace of $\mathbb{A}$. The dimension of L is defined as the dimension of its associated vector subspace: $\dim L = \dim \overline{L}$.

Notation Let $(\mathbb{A}, V, \phi)$ be a real affine space of dimension n. The subspaces of dimension 0 are the points of $\mathbb{A}$. The subspaces of dimension $1, 2$ and $n-1$ are called lines, planes and hyperplanes, respectively.

1.3.1. Equations of an affine subspace

Let $(\mathbb{A}, V, \phi)$ be an affine subspace with affine coordinate system $\mathcal{R} = \{O; B\}$, $B = \{e_1, \ldots, e_n\}$. Let $L \subset \mathbb{A}$ be an affine subspace of $\mathbb{A}$ of dimension k; this is, $L = P + \overline{L}$ with $\overline{L} = \langle u_1, \ldots, u_k \rangle$.

Suppose $(a_1, \ldots, a_n)$ are the coordinates of a point P in the coordinate system $\mathcal{R}$ and

$$\begin{cases} u_1 = a_{11}e_1 + \cdots + a_{n1}e_n \\ u_2 = a_{12}e_1 + \cdots + a_{n2}e_n \\ \qquad\vdots \\ u_k = a_{1k}e_1 + \cdots + a_{nk}e_n \end{cases}$$

Parametric equations

A point $X(x_1, \ldots, x_n)_{\mathcal{R}} \in L$ if and only if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$\overline{OX} = \overline{OP} + \lambda_1 u_1 + \cdots + \lambda_k u_k;$$

this is,

$$(x_1, \ldots, x_n) = (a_1, \ldots, a_n) + \lambda_1(a_{11}, \ldots, a_{n1}) + \cdots + \lambda_k(a_{1k}, \ldots, a_{nk})$$

or, equivalently

$$\left\{ \begin{array}{c} x_1 = a_1 + \lambda_1 a_{11} + \cdots + \lambda_k a_{1k} \\ \vdots \\ x_n = a_n + \lambda_1 a_{n1} + \cdots + \lambda_k a_{nk} \end{array} \right.$$

which are the parametric equations of the subspace L.

Cartesian equations

A point $X(x_1, \dots, x_n)_{\mathcal{R}} \in L$ if and only if the vector

$$\overline{PX} = (x_1 - a_1, \dots, x_n - a_n) \in \langle u_1, \dots, u_k \rangle \,.$$

As we are assuming that the vectors $u_1, \dots, u_k$ are linearly independent (if they were not, we would remove those which are a linear combination of the rest) we have

$$\operatorname{rank} \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix} = k.$$

Therefore, $\overline{PX} = (x_1 - a_1, \dots, x_n - a_n) \in \langle u_1, \dots, u_k \rangle$ if and only if

$$\operatorname{rank} \begin{pmatrix} x_1 - a_1 & a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ x_n - a_n & a_{n1} & \cdots & a_{nk} \end{pmatrix} = k.$$

As we are imposing the rank to be k we obtain $n-k$ minors of order $k+1$. This is, we obtain $n-k$ equations with n unknowns: $(x_1, \dots, x_n)$.

Observation Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B\}$, $B = \{e_1, \ldots, e_n\}$. Let

$$L \equiv \begin{cases} a_{11}x_1 + \cdots + a_{n1}x_n = b_1 \\ \qquad\qquad \vdots \\ a_{1r}x_1 + \cdots + a_{nr}x_n = b_r \end{cases}$$

be the cartesian equations of an affine subspace L of dimension $n - r$.

Notice that the cartesian equations of an affine subspace L of dimension $n - r$ are a system of r non homogeneous linear equations.

If $P, Q \in L$ then the vector $u = \overline{PQ}$ satisfies the equations of the homogeneous linear system associated with $\overline{L}$.

Proof of the previous claim If $P(p_1, \ldots, p_n)$ and $Q(q_1, \ldots, q_n)$ then

$$u = \overline{PQ} = (q_1 - p_1, \ldots, q_n - p_n)$$

and for $i = 1 \ldots r$, we have

$$\begin{aligned} & a_{1i}(p_1 - q_1) + \cdots + a_{ni}(p_1 - q_1) \\ = \; & a_{1i}p_1 + \cdots + a_{ni}p_n - (a_{1i}q_1 + \cdots + a_{ni}q_n) \underset{P,Q \in L}{=} b_i - b_i = 0. \end{aligned}$$

So, the system of cartesian equations of the vector space associated with L are:

$$\overline{L} \equiv \begin{cases} a_{11}x_1 + \cdots + a_{n1}x_n = 0 \\ \quad \vdots \\ a_{1r}x_1 + \cdots + a_{nr}x_n = 0 \end{cases}$$

Equations of a line

Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B\}$, $B = \{e_1, \ldots, e_n\}$. An affine line $r \subset \mathbb{A}$ is an affine subspace of dimension 1; this is, $r = P + \langle u \rangle$, with $P \in \mathbb{A}$ and $u \in V$. Let us suppose that $(a_1, \ldots, a_n)$ are the coordinates of a point P in the coordinate system $\mathcal{R}$ and

$$u = u_1 e_1 + \cdots + u_n e_n.$$

So, a point $X \in r$ if and only if

$$\overline{OX} = \overline{OP} + \lambda u,$$

this is, if $(x_1, \ldots, x_n)$ are the coordinates of X in the coordinate system $\mathcal{R}$ then,

$$(x_1, \ldots, x_n) = (a_1, \ldots, a_n) + \lambda(u_1, \ldots, u_n)$$

or, equivalently we obtain the parametric equations of the line r.

$$\begin{cases} x_1 = a_1 + \lambda_1 u_1 \\ \vdots \\ x_n = a_n + \lambda_1 u_n \end{cases}$$

If we assume $u_i \neq 0$, $i = 1, \ldots, n$ the former system is written as follows:

$$\frac{x_1 - a_1}{u_1} = \cdots = \frac{x_n - a_n}{u_n}$$

which is the continuous equation of the line r.
To finish, $X(x_1, \ldots, x_n)_{\mathcal{R}} \in L$ if and only if $\overline{XP} \in \langle u \rangle$ if and only if $\overline{XP}$ and u are proportional. Therefore, $\overline{XP} \in \langle u \rangle$ if and only if

$$\operatorname{rank} \begin{pmatrix} x_1 - a_1 & u_1 \\ \vdots & \vdots \\ x_n - a_n & u_n \end{pmatrix} = 1.$$

As we are imposing the rank to be 1 we obtain $n-1$ minors of order 2. This is, we obtain $n-1$ cartesian equations of r.

Equations of a hyperplane

Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B\}$, $B = \{e_1, \ldots, e_n\}$. An affine hyperplane $H \subset \mathbb{A}$ in an affine subspace of dimension $n-1$; it is therefore given by just one equation

$$a_1 x_1 + \cdots + a_n x_n = b.$$

Observation An affine subspace L of dimension k is the intersection of $n-k$ independent hyperplanes.

Example 1 Obtain the parametric equations of the affine subspace L of $\mathbb{A}$ which has the following cartesian equations with respect to $\mathcal{R} = \{O, \{e_1, e_2, e_3\}\}$:

$$L \equiv \begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_2 - x_3 = 1 \end{cases}$$

First method.

We solve the non homogeneous linear system of equations defining L. The coefficient matrix of the system is:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

whose rank is 2. As

$$\operatorname{rank} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2$$

if we take $x_3 = \lambda$ the system can be written as follows:

$$\begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_2 - x_3 = 1 \end{cases} \Longrightarrow \begin{cases} x_1 + x_2 = 1 - 2\lambda \\ 2x_2 = 1 + \lambda \\ x_3 = \lambda \end{cases}$$

This is,

$$\begin{cases} x_1 = \frac{1}{2} - \frac{5}{2}\lambda \\ x_2 = \frac{1}{2} + \frac{1}{2}\lambda \\ x_3 = \lambda \end{cases}$$

which are the parametric equations of L.

Second method.

As $\dim L = 3 - \operatorname{rank}(A) = 3 - 2 = 1$, L is a line, to determine L it is enough to give a point $P \in L$ and a vector v that generates the vector subspace $\overline{L} = \langle v \rangle$. The coordinates of a point $P \in L$ must satisfy the system of equations defined by L; for example take $P(3, 0, -1)_{\mathcal{R}}$.

The coordinates of a vector that generates the vector subspace $\overline{L}$ are a nontrivial solution of the homogeneous linear system:

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$$

For example $(-5, 1, 2)$ are the coordinates of the vector $u = -5e_1 + e_2 + 2e_3$. Therefore, $L = P + \overline{L} = P + \langle u \rangle$. So, $X(x_1, x_2, x_3)_{\mathcal{R}} \in L$ if and only if $(x_1, x_2, x_3) = (3, 0, -1) + \lambda(-5, 1, 2)$; this is,

$$\begin{cases} x_1 = 3 - 5\lambda \\ x_2 = \lambda \\ x_3 = -1 + 2\lambda \end{cases}$$

which are the parametric equations of L.

Example 2

Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B\}$, $B = \{e_1, e_2, e_3\}$. Obtain the cartesian equations of the affine subspace $L = P + \overline{L}$, where $P(1, 2, -1)_{\mathcal{R}}$ and $\overline{L} = \langle u_1, u_2 \rangle$ with $u_1 = e_1 + 2e_2 - e_3)$ and $u_2 = 2e_1 + e_2 + e_3$.

Solution.

The vectors u_1, u_2 which generate $\overline{L}$ are linearly independent. Therefore, $\dim L = 2$.

A point $X(x_1, x_2, x_3)_{\mathcal{R}} \in L$ if and only if the vector

$$\overline{PX} = (x_1 - 1, x_2 - 2, x_3 + 1) \in \langle u_1, u_2 \rangle \,;$$

this is, if and only if

$$\operatorname{rank} \begin{pmatrix} x_1 - 1 & 1 & 2 \\ x_2 - 2 & 2 & 1 \\ x_3 + 1 & -1 & 1 \end{pmatrix} = 2 \Longleftrightarrow 0 = \begin{vmatrix} x_1 - 1 & 1 & 2 \\ x_2 - 2 & 2 & 1 \\ x_3 + 1 & -1 & 1 \end{vmatrix} = 3x_1 - 3x_2 - 3x_3.$$

Therefore $L \equiv x_1 - x_2 - x_3 = 0$.

1.3.2 Intersection and sum of affine subspaces

Let $(\mathbb{A}, V, \phi)$ be a real affine space and L_1, L_2 two affine subspaces of $\mathbb{A}$. The intersection set of L_1 and L_2:

$$L_1 \cap L_2 = \{P \mid P \in L_1 \text{ and } P \in L_2\}$$

is an affine subspace of $\mathbb{A}$. If the intersection is not empty, $L_1 \cap L_2 \neq \emptyset$, then

$$\overline{L_1 \cap L_2} = \overline{L}_1 \cap \overline{L}_2.$$

We define the sum of L_1 and L_2 as the smallest affine subspace that contains L_1 and L_2 and it is denotated by $L_1 + L_2$. If $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ then

$$L_1 + L_2 = P_1 + \overline{L}_1 + \overline{L}_2 + \left\langle \overline{P_1 P_2} \right\rangle .$$

Two linear subspaces $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ intersect if and only if

$$\overline{P_1P_2} \in \overline{L}_1 + \overline{L}_2.$$

Observation If $L_1 \cap L_2 \neq \emptyset$ then

$$\overline{L_1 + L_2} = \overline{L}_1 + \overline{L}_2 + \left\langle \overline{P_1P_2} \right\rangle = \overline{L}_1 + \overline{L}_2,$$

If $L_1 \cap L_2 = \emptyset$ then

$$\overline{L_1 + L_2} = \overline{L}_1 + \overline{L}_2 + \left\langle \overline{P_1P_2} \right\rangle, \ P_1 \in L_1, \ P_2 \in L_2.$$

1.3.3 Parallelism

We say that two affine subspaces $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ of an affine space $(\mathbb{A}, V, \phi)$ are parallel if $\overline{L}_1 \subset \overline{L}_2$ or $\overline{L}_2 \subset \overline{L}_1$.
Two affine subspaces $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ may not intersect and they may not be parallel either, then they are skew subspaces.

1.3.4 Dimension Formula

Let $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ be two affine subspaces of an affine space $(\mathbb{A}, V, \phi)$. The following statements hold:

1. If $L_1 \cap L_2 \neq \emptyset$, then

$$\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(L_1 \cap L_2).$$

2. If $L_1 \cap L_2 = \emptyset$, then

$$\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(\overline{L}_1 \cap \overline{L}_2) + 1.$$

Example

Let $L_1 = P_1 + \overline{L}_1$ and $L_2 = P_2 + \overline{L}_2$ be two affine lines in an affine space $(\mathbb{A}, V, \phi)$ of dimension n. The possible relative positions of L_1 and L_2 are:
If $L_1 \cap L_2 \neq \emptyset$ then either $L_1 \cap L_2$ is a line and then $\dim(L_1 \cap L_2) = 1$ or $L_1 \cap L_2$ is a point and therefore $\dim(L_1 \cap L_2) = 0$. We have:

$$\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(L_1 \cap L_2)$$

L_1 and L_2 are coincident
L_1 and L_2 intersect in on point
$$\Longrightarrow \begin{cases} 1 = 1 + 1 - 1 \\ 2 = 1 + 1 - 0 \end{cases}$$

If $L_1 \cap L_2 = \emptyset$ then $\overline{L}_1 \cap \overline{L}_2$ can either be a vector line or the null vector $\overline{0}$. We have:

$$\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(\overline{L}_1 \cap \overline{L}_2) + 1$$

L_1 and L_2 are parallel
L_1 and L_2 are skew lines
$$\Longrightarrow \begin{cases} 2 = 1 + 1 - 1 + 1 \\ 3 = 1 + 1 - 0 + 1 \end{cases}$$

Definition

let $L_1 = P_1 + \langle u_1 \rangle$ and $L_2 = P_2 + \langle u_2 \rangle$ be two affine lines in an affine space $(\mathbb{A}, V, \phi)$ of dimension n. The following statements hold:

1. The lines L_1 and L_2 are skew lines if there does not exist a plane containing both lines; this is, if the vector system $\{u_1, u_2, \overline{P_1P_2}\}$ is linearly independent.
2. The lines L_1 and L_2 are in the same plane if they are not skew lines; this is, if the vector system $\{u_1, u_2, \overline{P_1P_2}\}$ is linearly dependent.
3. The lines L_1 and L_2 intersect if $L_1 \cap L_2 \neq \emptyset$.
4. The lines L_1 and L_2 are parallel if $\overline{L_1} = \overline{L_2}$; this is, if u_1 and u_2 are proportional. If besides $L_1 \cap L_2 \neq \emptyset$ then the two lines are coincident.

To study the systems of equations of two subspaces is a simple way of studying the relative position between those subspaces. We are going to study two particularly simple cases:

I. Relative position of two hyperplanes

Let $H_1, H_2 \subset \mathbb{A}$ be two hyperplanes with cartesian equations

$$\begin{aligned} H_1 &\equiv a_1x_1 + \cdots + a_nx_n = b, \\ H_2 &\equiv a'_1x_1 + \cdots + a'_nx_n = b'. \end{aligned}$$

The cartesian equations of their respective vector spaces are

$$\begin{aligned} \overline{H}_1 &\equiv a_1x_1 + \cdots + a_nx_n = 0, \\ \overline{H}_2 &\equiv a'_1x_1 + \cdots + a'_nx_n = 0. \end{aligned}$$

Therefore, if there exists λ such that $(a'_1, \ldots, a'_n) = \lambda\,(a_1, \ldots, a_n)$ then $\overline{H}_1 = \overline{H}_2$ and the hyperplanes H_1, H_2 are parallel.

If besides, $b' = \lambda b$ then the hyperplanes H_1, H_2 are coincident.

If $b' \neq \lambda b$ then the hyperplanes H_1, H_2 do not intersect ($H_1 \cap H_2 = \emptyset$).

II. Relative position between a line and a hyperplane

Let $(\mathbb{A}, V, \phi)$ be an affine space with affine coordinate system $\mathcal{R} = \{O; B\}$, $B = \{e_1, \ldots, e_n\}$. Let $r = P + \langle u \rangle$ be an affine line in $\mathbb{A}$ with $P(a_1, \ldots, a_n)_{\mathcal{R}}$ and $u = u_1 e_1 + \cdots + u_n e_n$. Let H be an affine hyperplane with cartesian equation

$$a_1 x_1 + \cdots + a_n x_n = b.$$

The line r and the hyperplane H are parallel if the vector $u \in \overline{H}$; this is, if $(u_1, \ldots, u_n)$ satisfies the homogeneous linear equation of the vector subspace $\overline{H}$; this is, if

$$a_1 u_1 + \cdots + a_n u_n = 0.$$