

CHAPTER II: AFFINE AND EUCLIDEAN GEOMETRY

4. ISOMETRIES

Definition

Let (\mathbb{E}, E, ϕ) and (\mathbb{E}', E', ϕ') be two euclidean affine spaces. An affine transformation $f: \mathbb{E} \longrightarrow \mathbb{E}'$ is an *isometry* if

$$d'(f(P), f(Q)) = d(P, Q), \quad \forall P, Q \in \mathbb{E},$$

where d is the distance defined in \mathbb{E} and d' is the distance defined in \mathbb{E}' .

Observation Isometries are always injective, because if $f(P) = f(Q)$ then

$$0 = d(f(P), f(Q)) = d(P, Q)$$

implies $P = Q$.

Proposition

An affine transformation $f: \mathbb{E} \longrightarrow \mathbb{E}'$ is an isometry if and only if its associated linear transformation $\bar{f}: E \longrightarrow E'$ preserves the dot product (this is, \bar{f} is a vector isometry).

Vector isometries are also called orthogonal endomorphisms.

Some properties of vector isometries Let $\bar{f}: E \longrightarrow E'$ be a vector isometry.

1. For all $\bar{u} \in E$ then $\|\bar{u}\| = \|\bar{f}(\bar{u})\|$.

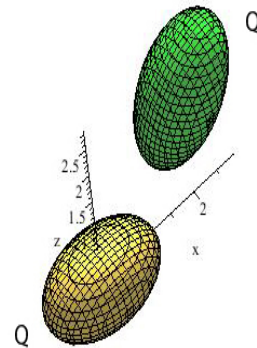
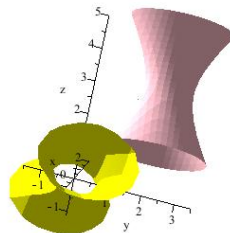
2. For all $\bar{u}, \bar{v} \in E$ then

$$\cos(\widehat{\bar{u}, \bar{v}}) = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \|\bar{v}\|} = \frac{\bar{f}(\bar{u}) \cdot \bar{f}(\bar{v})}{\|\bar{f}(\bar{u})\| \|\bar{f}(\bar{v})\|} = \cos(\widehat{\bar{f}(\bar{u}), \bar{f}(\bar{v})}).$$

3. If λ is a real eigenvalue of \bar{f} then $\lambda = \pm 1$.

4. If 1 and -1 are eigenvalues of \overline{f} then the eigenspaces V_1 and V_{-1} are orthogonal subspaces.
5. Let $E = E'$ with dimension n , the matrix A associated to \overline{f} in an orthonormal basis of E is an orthogonal matrix, that is $A^{-1} = A^t$ or equivalently $AA^t = I_n$.

Definition A *movement* is an isometry from an euclidean affine space \mathbb{E} onto itself.



4.1 Classification of isometries

The linear transformation \bar{f} associated to a movement $f: \mathbb{E} \longrightarrow \mathbb{E}$, is orthogonal. Therefore, in an orthonormal coordinate system $\mathcal{R} = \{O, B\}$, the matrix associated to f has the form:

$$M_f(R) = \begin{pmatrix} 1 & \bar{0}^t \\ \overline{Of(O)} & A \end{pmatrix}$$

where $A = M_{\bar{f}}(B)$ is an orthogonal matrix, this is, $A^{-1} = A^t$. Therefore, $\det A = \pm 1$.

If $\det A = 1$ we say that the isometry is a *direct* isometry.

If $\det A = -1$ we say that the isometry is an *indirect* isometry.

4.1.1 Isometries in the affine euclidean plane

Let f be an isometry from an euclidean affine space \mathbb{E} of dimension 2 on itself. Let $\mathcal{R} = \{O, B = \{\bar{e}_1, \bar{e}_2\}\}$ be an orthonormal coordinate system in \mathbb{E} . The matrix associated to f with respect to \mathcal{R} is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & \bar{0}^t \\ \bar{b} & A \end{pmatrix}, \text{ with } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } \bar{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The characteristic polynomial of A is $\det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$.

Subspace of fixed points The equation of the subspace of fixed points of f is

$$(A - I)X + \bar{b} = \bar{0}.$$

Therefore, f has fixed points if the former equation has a solution.

If $\text{rank}(A - I) = 2$ (also $\text{rank}(A - I|\bar{b}) = 2$) then f has only one fixed point.

If $\text{rank}(A - I) = \text{rank}(A - I|\bar{b}) = 1$ then f has a line of fixed points.

If $\text{rank}(A - I) = \text{rank}(A - I|\bar{b}) = 0$ then f is the identity transformation.

If $\det A = 1$, the isometry f is direct and $A \in SO(2)$ (matrices of order 2, orthogonal and with determinant 1). There exists an angle θ such that

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Notice that, in this case, $\det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + 1$ and $\operatorname{tr}(A) = 2 \cos \theta$.

1. If $\cos \theta = \frac{1}{2}\operatorname{tr}(A) \neq 1$, then $\lambda = 1$ is not an eigenvalue of the matrix A and, therefore $\operatorname{rank}(A - I) = 2$ and f has only one fixed point that we denote by P . In this case, f is a **rotation of angle θ and with center in the fixed point P** . In the coordinate system $\mathcal{R}' = \{P, B = \{\bar{e}_1, \bar{e}_2\}\}$ the matrix associated to f is

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}.$$

If $\cos \theta = \frac{1}{2}\operatorname{tr}(A) = -1$ then $\theta = 180^\circ$ and f is a **central symmetry with center in the fixed point P** .

2. If $\cos \theta = \frac{1}{2} \text{tr}(A) = 1$, then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and f is a *translation of vector* \bar{b} .

- a) $\text{rank}(A - I) = \text{rank}(A - I|\bar{b}) = 0$ then f is the **identity transformation**.
- b) $\text{rank}(A - I) \neq \text{rank}(A - I|\bar{b})$ then f is the **translation of vector** \bar{b} .

If $\det(A) = -1$, the isometry f is indirect and $A \in O(2)$ (orthogonal matrices of order 2). The eigenvalues of A are 1, -1 . If we take \bar{u}_1 eigenvector associated to 1 and \bar{u}_2 eigenvector associated to -1 , we have that in the basis $B' = \{\bar{u}_1, \bar{u}_2\}$ the matrix associated to \bar{f} (which we keep on calling A) is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $\text{rank}(A - I) = 1$.

1. If $\text{rank}(A - I|\bar{b}) = 1$ then there exists a line of fixed points of f . Let P be a point of this line (this is, a fixed point of f), in the orthonormal coordinate system $\mathcal{R}' = \left\{ P, \left\{ \frac{1}{\|\bar{u}_1\|} \bar{u}_1, \frac{1}{\|\bar{u}_2\|} \bar{u}_2 \right\} \right\}$ the matrix associated to f is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and f is an **axial symmetry**. The line of fixed points $r \equiv P + \langle \bar{u}_1 \rangle$ is called *axis of the symmetry*.

2. If $\text{rank}(A - I|\bar{b}) = 2$ then f has no fixed points. In the orthonormal coordinate system $\mathcal{R}' = \left\{ O, \left\{ \frac{1}{\|\bar{u}_1\|} \bar{u}_1, \frac{1}{\|\bar{u}_2\|} \bar{u}_2 \right\} \right\}$ the matrix associated to f is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \\ c_2 & 0 & -1 \end{pmatrix}.$$

Let us study whether in this case there exists any invariant line. We know that $V_1 = \langle \bar{u}_1 \rangle$ and $V_{-1} = \langle \bar{u}_2 \rangle$. Let us compute $\overline{Xf(X)}$. Let (x', y') be the coordinates in the coordinate system \mathcal{R}' of an arbitrary point X , we have:

$$\begin{aligned} \overline{Xf(X)} &= f(X) - X = (x' + c_1, -y' + c_2) - (x', y') \\ &= (c_1, -2y' + c_2). \end{aligned}$$

If $-2y' + c_2 = 0$ then $\overline{Xf(X)} \in \langle \bar{u}_1 \rangle$. Therefore, the line with equation $-2y' + c_2 = 0$ is an invariant line of f . If we take a point P as the origin of the coordinate system in the above mentioned line (then the coordinates of P have the form $(p, \frac{c_2}{2})$), we have that in the coordinate system $\mathcal{R}' = \left\{ P, \left\{ \frac{1}{\|\bar{u}_1\|} \bar{u}_1, \frac{1}{\|\bar{u}_2\|} \bar{u}_2 \right\} \right\}$ the matrix of f is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This is the composition of an axial symmetry, with axis the invariant line $P + \langle \bar{u}_1 \rangle$, and a translation parallel with the axis (with vector $(p, 0)$).

Observation. Every symmetry composed with a translation can be decomposed in an unique way as a symmetry composed with a translation with the direction vector of the axis as its vector.

Table of classification

$$\det A = 1 \text{ (then } \cos \alpha = \frac{1}{2} \text{tr} A \text{)}$$

	$\text{rank}(A - I)$	$\text{rank}(A - I \mid \bar{b})$	Classification
$\cos \alpha = 1$	0	0 ($\bar{b} = \bar{0}$)	Identity
$\cos \alpha = 1$	0	1 ($\bar{b} \neq \bar{0}$)	Translation
$\cos \alpha \neq 1$	2	2	Rotation

$$\det A = -1$$

$\text{rank}(A - I)$	$\text{rank}(A - I \mid \bar{b})$	Classification
1	1	Symmetry with respect to line of fixed points
1	2	Translational symmetry

Example

Classify the isometry $f(x, y) = (1 - y, 3 - x)$.

Solution

The matrix associated with this isometry is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}, \text{ we denote by } A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \text{ and } \bar{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

As $\det(A) = -1$ the isometry is indirect, has eigenvalues $\lambda = -1, 1$ and, in this case, $\bar{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is an unitary eigenvector associated to the eigenvalue $\lambda = -1$ and $\bar{e}_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is an unit eigenvector associated with the eigenvalue $\lambda = 1$. Let us see if f has fixed points. As

$$\text{rank}(A - I) = 1 \text{ and } \text{rank}(A - I|\bar{b}) = 2$$

the isometry f has no fixed points.

It is an isometry composed with a translation. Let us check if it has any invariant line, and let us calculate it

$$\begin{aligned}\overline{Xf(X)} &= f(X) - X = (1 - y, 3 - x) - (x, y) \\ &= (1 - x - y, 3 - x - y) \in V(-1) \text{ or } V(1)\end{aligned}$$

Then, $\overline{Xf(X)} \in V(-1)$ if and only if $\overline{Xf(X)}$ and \bar{e}_1 are proportional; this is, if

$$\begin{aligned}1 - x - y &= t \\ 3 - x - y &= t\end{aligned}$$

Subtracting both equations we obtain $2 = 0$ which is impossible.

And $\overline{Xf(X)} \in V(1)$ if and only if $\overline{Xf(X)}$ and \bar{e}_2 are proportional; this is, if

$$\begin{aligned}1 - x - y &= -t \\ 3 - x - y &= t\end{aligned}$$

Subtracting both equations we obtain $t = 1$ and therefore, $\overline{Xf(X)} \in V(1)$ if and only if

$$x + y = 2$$

which is the equation of the invariant line.

Therefore, f is a translational symmetry; this is, a symmetry s with the invariant line as axis, composed with a translation with vector proportional to the eigenvector associated to the eigenvalue $\lambda = 1$ (direction vector of the invariant line). The matrix of the symmetry is

$$M_s(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ a & 0 & -1 \\ b & -1 & 0 \end{pmatrix}$$

where a, b let us fix any point of the line $x+y = 2$. For example, let us impose it fixes the point $(1, 1)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 0 & -1 \\ b & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies \begin{cases} a = 2 \\ b = 2 \end{cases}$$

Let us compute which is the translation vector:

$$\begin{aligned} M_f(\mathcal{R}) &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ v_1 + 2 & 0 & -1 \\ v_2 + 2 & -1 & 0 \end{pmatrix} \end{aligned}$$

then $v_1 = -1$ and $v_2 = 1$.

We will construct the matrix $M_f(\mathcal{R})$ of any isometry $f: E_2 \rightarrow E_2$ from the matrices of the next basic isometries:

Translation f of vector $v = (a, b)$:

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}.$$

Symmetry f :

Axis (line of fixed points)	$x = 0$	$y = 0$
$M_f(\mathcal{R})$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

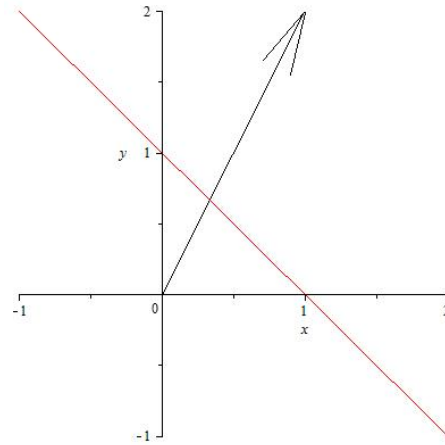
Rotation f of angle α and center the origin:

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ a \cos(\alpha) & -\sin(\alpha) & \\ b \sin(\alpha) & \cos(\alpha) & \end{pmatrix}.$$

Example Obtain the analytic expression of the isometry of the plane which is a composition of the symmetry with the line $x + y = 1$ as axis and the translation with vector $\bar{v} = (1, 2)$. Decompose the obtained isometry as composition of a symmetry and a translation with vector parallel with the symmetry axis.

Solution The vector line associated to the symmetry axis has cartesian equation $x + y = 0$.

Let us consider the coordinate system $\mathcal{R}' = \{P, \{\bar{u}_1, \bar{u}_2\}\}$ where P is a point of the symmetry axis, for example $P(1, 0)$, the vector \bar{u}_1 is an unitary vector in the line $x + y = 0$; for example $\bar{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and vector \bar{u}_2 is an unitary vector orthogonal to \bar{u}_1 ; this is, $\bar{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.



In the mentioned coordinate system the matrix associated to the symmetry S with axis $x + y = 1$ is

$$M_S(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} M_S(\mathcal{R}) &= M(\mathcal{R}', \mathcal{R})M_S(\mathcal{R}')M(\mathcal{R}, \mathcal{R}') \\ &= M(\mathcal{R}', \mathcal{R})M_S(\mathcal{R}')M(\mathcal{R}', \mathcal{R})^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \end{aligned}$$

The translation T with vector $\bar{v} = (1, 2)$ has associated matrix:

$$M_T(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Thus, the matrix associated to the isometry we are looking for, is

$$\begin{aligned} M_{T \circ S}(\mathcal{R}) &= M_T(\mathcal{R})M_S(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}. \end{aligned}$$

And

$$(T \circ S)(x, y) = (2 - y, 3 - x).$$

We are going to decompose the obtained isometry as a composition of a symmetry and a translation t_2 with vector parallel with the symmetry axis. Let us decompose the vector $\bar{v} = (1, 2)$ as an addition of a vector with direction parallel with the symmetry axis s and a vector orthogonal to this forementioned vector:

$$\bar{v} = (1, 2) = a(1, -1) + b(1, 1),$$

from where $a = -\frac{1}{2}$ and $b = \frac{3}{2}$. Therefore, let us take the translation t_2 with vector $\bar{v}_2 = (-\frac{1}{2}, \frac{1}{2})$. Let us calculate the symmetry s_2 :

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ c & 0 & -1 \\ d & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ c - \frac{1}{2} & 0 & -1 \\ d + \frac{1}{2} & -1 & 0 \end{pmatrix} \end{aligned}$$

from where, $c = \frac{5}{2}$ and $d = \frac{5}{2}$. Then,

$$M_{s_2}(\mathcal{R}) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 0 & -1 \\ \frac{5}{2} & -1 & 0 \end{pmatrix}.$$

Now let us compute the line of fixed points of the symmetry s_2 . We have:

$$\begin{aligned} \overrightarrow{X s_2(X)} &= \left(\frac{5}{2} - y, \frac{5}{2} - x \right) - (x, y) \\ &= \left(\frac{5}{2} - x - y, \frac{5}{2} - x - y \right) \\ &= \left(\frac{5}{2} - x - y \right) (1, 1). \end{aligned}$$

Therefore, the line $5 = 2x + 2y$ is the line of fixed points of the symmetry s_2 (is the symmetry axis).

4.2 Isometries in the tridimensional euclidean affine space

Let f be an isometry from an euclidean affine space \mathbb{E} of dimension 3 on itself. Let $\mathcal{R} = \{O, B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}\}$ be an orthonormal coordinate system in \mathbb{E} . The matrix associated to f with respect to \mathcal{R} is

$$M_f(\mathcal{R}) = \begin{pmatrix} 1 & \bar{0}^t \\ \bar{b} & A \end{pmatrix}, \text{ with } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } \bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The characteristic polynomial of A is $\det(A - \lambda I) = -\lambda^3 + tr_2(A)\lambda^2 - tr(A)\lambda + \det(A)$, where

$$tr_2(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

Subspace of fixed points

The equation of the subspace of fixed points of f is

$$(A - I)X + \bar{b} = \bar{0}.$$

Therefore, f has fixed points if the above mentioned equation has a solution.

If $\text{rank}(A - I) = 3$ (then also $\text{rank}(A - I|\bar{b}) = 3$) then f has only one fixed point.

If $\text{rank}(A - I) = \text{rank}(A - I|\bar{b}) = 2$ then f has a line of fixed points.

If $\text{rank}(A - I) = \text{rank}(A - I|\bar{b}) = 1$ then f has a plane of fixed points.

If $\text{rank}(A - I) = \text{rank}(A - I|\bar{b}) = 0$ then f is the identity transformation.

If $\det A = 1$, the isometry f is direct and $A \in SO(3)$ (orthogonal matrices of order 3 and with determinant 1) and, in a convenient orthonormal basis $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ the matrix associated to \bar{f} is written:

$$M_{\bar{f}}(B') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}.$$

Notice that, in this case, $tr(A) = 1 + 2 \cos \theta$.

1. If $\cos \theta = 1$, then $\text{rank}(A - I) = 0$ and we can encounter two situations:

a) $\text{rank}(A - I|\bar{b}) = 0$ and, in this case,

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and f is the identity transformation.

b) $\text{rank}(A - I|\bar{b}) = 1$, in this case there are no fixed points and f is a translation of vector \bar{b} . The matrix associated with f in this case is:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix}.$$

2. If $|\cos \theta| \neq 1$ then $\text{rank}(A - I) = 2$ and we can encounter two situations:

a) $\text{rank}(A - I|\bar{b}) = 2$ and, in this case, there is a line of fixed points $r \equiv Q + \langle \bar{w}_1 \rangle$, where \bar{w}_1 is an eigenvector associated with the eigenvalue $\lambda = 1$. In the coordinate system

$$\mathcal{R}' = \left\{ Q, \left\{ \bar{u}_1 = \frac{1}{\|\bar{w}_1\|} \bar{w}_1, \bar{u}_2, \bar{u}_3 \right\} \right\}$$

the matrix associated to f is

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}.$$

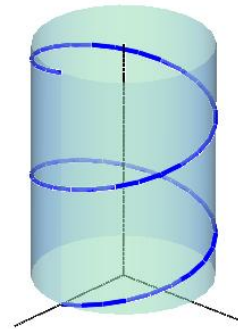
Thus f is a rotation of angle θ and axis the line r of fixed points.

In the particular case where $\cos \theta = -1$, we would have an axial symmetry with the line r of fixed points as axis.

b) $\text{rank}(A - I|\bar{b}) = 3$ and, in this case, there are no fixed points. The matrix associated with f can be written as follows:

$$\begin{aligned}
 M_f(\mathcal{R}') &= \begin{pmatrix} 1 & \bar{0}^t \\ \bar{b} & A \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}
 \end{aligned}$$

and f is an **helical movement**, this is a rotation of angle θ and axis the invariant line f , with associated vector subspace V_1 , compounded with a translation parallel with the fore mentioned line (with vector $\bar{u} = \overline{Xf(X)}$, where $X \in r$).



If $\det A = -1$, the isometry f is indirect and $A \in O(3)$ (orthogonal matrices of order 3).

In a convenient orthonormal basis $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$, the unitary vector \bar{u}_1 is the eigenvector associated with $\lambda = -1$, the matrix associated with \bar{f} is written as follows:

$$M_{\bar{f}}(B') = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}.$$

Notice that, in this case, $tr(A) = -1 + 2 \cos \theta$.

1. If $\cos \theta = 1$ then $\text{rank}(A - I) = 1$.

a) If $\text{rank}(A - I|\bar{b}) = 1$ then there exists a plane of fixed points $\pi \equiv P + \langle \bar{v}_1, \bar{v}_2 \rangle$. In the coordinate system

$$\mathcal{R}' = \left\{ Q, \left\{ \bar{u}_1, \bar{u}_2 = \frac{1}{\|\bar{v}_1\|} \bar{v}_1, \bar{u}_3 = \frac{1}{\|\bar{v}_2\|} \bar{v}_2 \right\} \right\}$$

the matrix associated to f is written as follows

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and f is a **specular symmetry (or reflection)** with respect to the plane of fixed points.

b) If $\text{rank}(A - I|\bar{b}) = 2$ then there are no fixed points. The matrix associated with f can be written as follows:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & \bar{0}^t \\ \bar{b} & A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and f is a **symmetry composed with a translation with vector parallel to the invariant plane.**

2. If $\cos \theta \neq 1$ then \bar{f} does not have the eigenvalue $\lambda = 1$ and there is an unique fixed point Q . In the orthonormal coordinate system $\mathcal{R}' = \{Q, \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}\}$ the matrix associated with f is written:

$$M_f(\mathcal{R}') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}$$

and f is a **symmetry (with respect to the plane $Q + \langle \bar{u}_2, \bar{u}_3 \rangle$) compounded with a rotation of angle θ and axis $Q + \langle \bar{u}_1 \rangle$.**

In the particular case where $\cos \theta = -1$, then f is a **central symmetry with the only fixed point Q as the center.**

Table of classification

$$\det A = 1; \cos \alpha = \frac{1}{2}(\text{tr}(A - 1))$$

$\text{rank}(A - I)$	$\text{rank}(\bar{b} \mid A - I)$	Classification
0	0 ($\bar{b} = \bar{0}$)	Identity
0	1 ($\bar{b} \neq \bar{0}$)	Translation
2	2	Rotation of angle α
2	3	Helical movement

$$\det A = -1; \cos \alpha = \frac{1}{2}(\text{tr} A + 1)$$

$\text{rank}(A - I)$	$\text{rank}(\bar{b} \mid A - I)$	Classification
1	1	Symmetry
1	2	Translational symmetry
3	3	Composition of a rotation and a symmetry